Deformation Quantisation of Constrained Systems

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Abstract

We study the deformation quantisation (Moyal quantisation) of general constrained Hamiltonian systems. It is shown how second class constraints can be turned into first class quantum constraints. This is illustrated by the $O(N)$ non-linear σ-model. Some new light is also shed on the Dirac bracket. Furthermore, it is shown how classical constraints not in involution with the classical Hamiltonian, can be turned into quantum constraints *in* involution with respect to the Hamiltonian. Conditions on the existence of anomalies are also derived, and it is shown how some kinds of anomalies can be removed. The equations defining the set of physical states are also given.

It turns out that the deformation quantisation of pure Yang-Mills theory is straightforward whereas gravity is anomalous. A formal solution to the Yang-Mills quantum constraints is found. In the ADM formalism of gravity the anomaly is very complicated and the equations picking out physical states become infinite order functional differential equations, whereas the Ashtekar variables remedy both of these problems – the anomaly becoming simply a central extension (Schwinger term) and the equations for physical states become finite order.

We finally elaborate on the underlying geometrical structure and show the method to be compatible with BRST methods.

1 Introduction

The problem of how to quantise a given classical system is one of the oldest in the history of quantum theory, its age being comparable to that of the measurement problem. This being so, many attempts have been made in the past of defining general quantisation schemes. Usually such schemes try to construct mappings $q_i, p^j \mapsto \hat{q}_i, \hat{p}^j$ such that

$$\{f(q,p), g(q,p)\}_{\rm PB} \mapsto \frac{1}{i\hbar}[\hat{f}(\hat{q},\hat{p}), \hat{g}(\hat{q},\hat{p})] \tag{1}$$

where $\{\cdot,\cdot\}_{\rm PB}$ denotes the Poisson bracket,

$$\{f(q,p), g(q,p)\}_{\rm PB} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i} \tag{2}$$

and where $\hat{f}$ denotes some kind of operator-valued function obtained from the classical observable f in some prescribed manner. The most elaborate of these schemes must be geometric quantisation, [1], which relies heavily on the symplectic geometry of classical phasespaces.
Unfortunately, one can show that in general no quantisation procedure can exists for functions which are not at most quadratic polynomials in the basic variable, q_i, p^j, [1]. This implies that the algebra will receive quantum corrections. In fortunate circumstances these are just central extensions. [1]
Another procedure, which has not been studied as much as geometric quantisation, is *deformation quantisation*. It is known that this always works, [2] (recently, Kontsevich has extended the proof of existence from symplectic to Poisson manifolds too, [27]). Denote the classical phasespace by Γ, this is then a symplectic manifold, and the set of observables, $C^\infty(\Gamma)$, form a Poisson-Lie algebra (i.e., a Lie algebra under Poisson brackets), $\mathcal{A}$. Deformation quantisation consists in replacing the algebra of observables with another, deformed one, $\mathcal{A}_\hbar$, and the Poisson brackets by a new, deformed

[1] One should note that BRST does not escape this problem either - given a classical BRST generator, the task is yet again to find an operator realisation. The problem is lightened a little bit, however, since BRST exploits the structure of the classical theory to a far greater extent than old fashioned canonical quantisation. Ultimately, though, it ends up with similar problems.
Geometric quatisation avoids the no-go theorem by loosening another of Dirac's requirements, namely that q, p be irreducibly represented. This can lead to problems with the classical limit, however.

bracket. More concretely, $\mathcal{A}_\hbar = \mathcal{A} \otimes \mathbb{C}((\hbar))$, i.e., the elements of $\mathcal{A}_\hbar$ are functions on phasespace which can also be expanded in a power series in $\hbar, \hbar^{-1}$.[2] Thus elements of $\mathcal{A}_\hbar$ can be written as

$$f(q,p,\hbar) = ... + \hbar^{-1} f_{-1}(q,p) + f_0(q,p) + \hbar f_1(q,p) + ... \tag{3}$$

where $f_n \in \mathcal{A} = C^\infty(\Gamma), n \in \mathbb{Z}$. Here f_0 is the *classical part* of f. Similarly one writes the new bracket as

$$[f,g]_M = i\hbar\{f,g\}_{\mathrm{PB}} + O(\hbar^2) \tag{4}$$

We will in general refer to the deformed brackets as *Moyal brackets*, even though that name is strictly speaking reserved for deformed brackets of flat phasespaces, $\Gamma \simeq \mathbb{R}^{2n}$. It has recently been proven by Tzanakis and Dimakis, [3], that the Moyal bracket is essentially unique – the various possibilities corresponding to various operator ordering prescriptions. Consequently, deformation quantisation does not suffer from the usual amibiguities of other quantisation schemes. Furthermore, Dereli and Verçin have proven that the various possibilities for the Moyal bracket du to different operator orderings, satisfy a W_∞-covariance, [25]. The close relationship between W-symmetry and the Wigner-Weyl-Moyal formalism has also been extensively studied by Gozzi and coworkers, [26].
Furthermore, we will usually restrict ourselves to the simpler case of only non-negative powers of $\hbar$, i.e., work with $\mathcal{A}_\hbar = \mathcal{A} \otimes \mathbb{C}[[\hbar]]$. Only when this turns out to be impossible will we deal with the more general case of $\mathcal{A} \otimes \mathbb{C}((\hbar))$. Systems with second class constraints or anomalies will in general require the presence of negative powers of $\hbar$, whereas systems without such problems in general will not.
The general philosophy of the paper will be that for any classical theory with problems (such as second class constraints, Hamiltonian not in involution with the constraints, anomalies appearing upon a simple quantisation), a completely well-behaved quantum theory exists for which all of these problems are absent. The problems only reappear when taking the classical limit, a procedure which has to be slightly modified. In other words, we have a

[2] Standard mathematical notation: $\mathbb{C}[x]$ denotes the set of complex polynomials in one variable x, $\mathbb{C}[[x]]$ the set of formal power series in x, $\mathbb{C}(x)$, the field of fractions of $\mathbb{C}[x]$ (i.e., $\mathbb{C}(x) = \{f(x)/g(x) \mid f,g \in \mathbb{C}[x], g(x) \neq 0\}$, and $\mathbb{C}((x))$ the field of fractions of $\mathbb{C}[[x]]$, i.e., $\mathbb{C}((x)) \simeq \mathbb{C}[[x,x^{-1}]]$.

"quantum smoothening out" of classical problems, analogous to what happens in quantum chaos, where a chaotic classical Hamiltonian still leads to an integrable system at the quantum level, because the phasespace gets "smurred" by the Heisenberg uncertainty principle. I think this is a sensible procedure, as the quantum description is presumably the most fundamental one, and the classical description should at most be considered as a limiting case – and as such, we would expect problems to occur, not so much at the fundamental level (nature tends to favour simplicity) but arrising from taking the limit.
For $\Gamma \simeq \mathbb{R}^{2n}$ a compact expression for the Moyal bracket exists. Introduce the bidifferential operator $\triangle$,[3]

$$f \triangle g = \{f, g\}_{\mathrm{PB}} \tag{5}$$

then

$$[f, g]_M = 2if \sin(\frac{1}{2}\hbar\triangle)g \tag{6}$$

see e.g. [2]. This can be written out more explicitly as

$$[f, g]_M = i \sum_{n=0}^{\infty} \frac{(-1)^n \hbar^{2n+1}}{4^n(2n+1)!} \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \frac{\partial^{2n+1} f}{\partial q^{2n-k+1} \partial p^k} \frac{\partial^{2n+1} g}{\partial p^{2n-k+1} \partial q^k} \tag{7}$$

with

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

denoting a binomial coefficient.
The Moyal bracket can be written in terms of a so-called twisted product

$$f * g = f e^{\frac{1}{2} i\hbar\triangle} g = fg + O(\hbar) \tag{8}$$

the Moyal bracket is then the commutator with respect to this product

$$[f, g]_M = f * g - g * f \tag{9}$$

which brings out the analogy with the standard operator formulation of quantum theory.

[3] We will only in this paper deal with bosonic degrees of freedom. Fermionic variables can be treated by means of a $\mathbb{Z}_2$-grading, where the usual Poisson bracket is replaced by a graded analogue, as in the Batalin-Fradkin-Vilkovisky approach to BRST-quantisation.

In general we will write

$$[f,g]_M = \sum_{n=1} i\hbar^n \omega_n(f,g) \tag{10}$$

with $\omega_1(f,g) = \{f,g\}_{\text{PB}}$.
Since one can always choose local Darboux coordinates, q_i, p^j (with the standard Poisson bracket) we can always at least locally make do with the Moyal bracket, without having to work with a more general deformation. Furthermore, for most systems of physical interest, such as *all* standard field theories, we can take the classical phasespace to be "flat" in the sense that it can be covered by a global coordinate patch of Darboux coordinates. The possible "curvature" of Γ is put into a set of constraints. Hence one may extend the original phasespace, which might not have a global Darboux coordinate patch, into a "flat" phasespace Γ', such that Γ is characterised by the vanishing of certain functions $\phi_a(q,p)$, i.e.,

$$\Gamma = \{(q,p) \in \Gamma' \mid \forall a : \phi_a(q,p) = 0\} \tag{11}$$

This leads us naturally to study Hamiltonian systems with constraints.

2 Hamiltonian Systems with Constraints

Consider a classical Hamiltonian system with a phasespace Γ and a Hamiltonian $h(q,p)$. Suppose the transition from a Hamiltonian to a Lagrangian picture is singular, i.e., suppose not all the phasespace coordinates are independent. We then have a system with constraints, i.e., a family of functions $\phi_a(q,p)$ exists such that the true degrees of freedom are characterised by the vanishing of these functions. We will also assume these functions to be independent.
We will, at first, assume that the set of constraints are *first class*, i.e., they satisfy a Poisson-Lie algebra

$$\{\phi_a, \phi_b\}_{\text{PB}} = c_{ab}{}^c \phi_c \tag{12}$$

where the structure coefficients, $c_{ab}{}^c$ can be functions of the phasespace variables.
For now we will also assume that time evolution does not give rise to new constraints, i.e., the functions ϕ_a are *in involution* with the Hamiltonian

$$\{h, \phi_a\}_{\text{PB}} = V_a^b \phi_b \tag{13}$$

where again the structure coefficient V_a^b can depend on the phasespace coordinates. We will return to the more general setting in a subsection later.
Now we want to perform a deformation quantisation. We will thus replace ϕ_a by functions $\Phi_a = \phi_a + O(\hbar, \hbar^{-1})$, the Hamiltonian h with $H = h + O(\hbar, \hbar^{-1})$ and the Poisson brackets with Moyal ones, in such a manner that

$$[\Phi_a, \Phi_b]_M = i\hbar c_{ab}^c \Phi_c \tag{14}$$

$$[H, \Phi_a]_M = i\hbar V_a^b \Phi_b \tag{15}$$

For simplicity we will only consider Φ_a, H to be formal powerseries in $\hbar$ and not $\hbar^{-1}$. Write

$$\Phi_a = \phi_a + \sum_{n=1}^{\infty} \hbar^n \Phi_a^{(n)} \qquad H = h + \sum_{n=1}^{\infty} \hbar^n H_n \tag{16}$$

Furthermore we have already assumed that the structure coefficients are undeformed. When the Moyal bracket algebra, (14-15), become non-isomorphic to the classical Poisson bracket one we say we have an *anomaly.* This can happen if e.g. the structure constants get $\hbar$-corrections, the algebra become centrally extended or if the right hand sides of (14-15) become non-linear in the Φ_a. Such problems will be dealt with in a later section.
At each order in $\hbar$ the Moyal algebra relations (14- 15) read

$$\sum_{n,k} \omega_{N-n-k}(\Phi_a^{(n)}, \Phi_b^{(k)}) = c_{ab}^c \Phi_c^{(N)} \tag{17}$$

$$\sum_{n,k} \omega_{N-n-k}(H_n, \Phi_a^{(k)}) = V_a^b \Phi_b^{(N)} \tag{18}$$

which provides us with a recursive scheme for finding the quantum constraints Φ_a and the quantum Hamiltonian H. Inserting the explicit form for ω_l we get

$$\sum_{n,k} \sum_{l=0}^{N-n-k} \frac{(-1)^{N-n-k+l}}{4^{N-n-k}(2N-2n-2k+1)!} \binom{N-n-k}{l} \times \frac{\partial^{N-n-k} \Phi_a^{(n)}}{\partial p^{N-n-k-l} \partial q^l} \frac{\partial^{N-n-k} \Phi_b^{(k)}}{\partial q^{N-n-k-l} \partial p^l} = c_{ab}^c \Phi_c^{(N)} \tag{19}$$

$$\sum_{n,k} \sum_{l=0}^{N-n-k} \frac{(-1)^{N-n-k+l}}{4^{N-n-k}(2N-2n-2k+1)!} \binom{N-n-k}{l} \times$$

$$\frac{\partial^{N-n-k} H_n}{\partial p^{N-n-k-l}\partial q^l}\frac{\partial^{N-n-k}\Phi_a^{(k)}}{\partial q^{N-n-k-l}\partial p^l} = V_a^b \Phi_b^{(N)} \quad (20)$$

One should note that if ϕ_a, h are all at most third order in q, p then $\omega_n, n \geq 3$ on these vanish identically and one can take simply $\Phi_a = \phi_a, H = h$. Many physical systems have precisely this structure. Most notably the Yang-Mills field theory. The study of membranes or some supersymmetrical theories, however, will in general need $\hbar$-corrections to the constraints and Hamiltonian.

It is well worth having a closer look at the recursive scheme above. The first equation, the one for the first quantum correction, reads

$$c_{ab}^c \Phi_c^{(1)} = \{\phi_a, \Phi_b^{(1)}\}_{\text{PB}} + \{\Phi_a^{(1)}, \phi_b\}_{\text{PB}} \quad (21)$$

which gives a first order partial differential equation for $\Phi_a^{(1)}$,

$$\left[c_{ab}^c + \delta_a^c\left(\frac{\partial\phi_b}{\partial q}\frac{\partial}{\partial p} - \frac{\partial\phi_b}{\partial p}\frac{\partial}{\partial q}\right) - \delta_b^c\left(\frac{\partial\phi_a}{\partial q}\frac{\partial}{\partial p} - \frac{\partial\phi_a}{\partial p}\frac{\partial}{\partial q}\right)\right]\Phi_c^{(1)} = 0 \quad (22)$$

Introduce the operator

$$\mathcal{D}_{ab}^c = c_{ab}^c + \delta_a^c\{\phi_b, \cdot\}_{\text{PB}} - \delta_b^c\{\phi_a, \cdot\}_{\text{PB}} \quad (23)$$

we can then write the equation for $\Phi_c^{(1)}$ as

$$\mathcal{D}_{ab}^c \Phi_c^{(1)} = 0 \quad (24)$$

The other equations in the hierarchy have a similar form, namely

$$\mathcal{D}_{ab}^c \Phi_c^{(n)} = \mathcal{J}_{ab}^{(n)} \quad (25)$$

where $\mathcal{J}_{ab}^{(n)}$ is a source term only depending on $\Phi_c^{(0)} = \phi_c, \Phi_c^{(1)}, ..., \Phi_c^{(n-1)}$. Explicitly

$$\mathcal{J}_{ab}^{(n)} = -\sum_{m=1}^{n}\sum_{k=1}^{m+1}\omega_m(\Phi_a^{(k)}, \Phi_b^{(n+1-m-k)}) \quad (26)$$

which shows how the "source" is build up from the various m-order brackets, ω_m.

Note, by the way, that if (24) has only the trivial solution $\Phi_c^{(1)} \equiv 0$, then it follows from (25) that $\Phi_c^{(n)} \equiv 0, n \geq 1$ and hence $\Phi_a = \phi_a$. This would of

course not be the case if we included negative powers of $\hbar$ in the expansion of Φ_c. In any case equations (24-25) show that $\Phi_c^{(n)}$ cannot be uniquely defined as we can always add a multiple of $\Phi_c^{(1)}$ to it (with an appropriate factor of $\hbar$ in front, naturally).
Furthermore, the operator $\mathcal{D}_{ab}^c$ cannot be identically zero, since the classical constraints satisfy

$$\mathcal{D}_{ab}^c \phi_c = -c_{ab}{}^c \phi_c \tag{27}$$

and are hence "eigenvectors" of $\mathcal{D}_{ab}^c$. Equation (25) constitue a set of first order partial differential equations determining the quantities $\Phi_a^{(n)}$.
There is a cohomological element to this set of equations. Denote the algebra of constraints by $\mathfrak{g}$, $\mathfrak{g} \subseteq \mathcal{A} = C^\infty(\Gamma)$, and form the complex $\mathcal{C}^n = \wedge^n \mathfrak{g}$ of alternating n-linear expressions in the constraints ϕ_a. Introduce the operator $\hat{A}_{ab}{}^c : \mathcal{C}^1 \to \mathcal{C}^2$ by

$$\hat{A}_{ab}{}^c f_c := \{\phi_a, f_b\}_{\mathrm{PB}} - \{\phi_b, f_a\}_{\mathrm{PB}} \tag{28}$$

We extend this by linearity to a map $\mathcal{C}^p \to \mathcal{C}^{p+1}$,

$$\hat{A}_{ab}{}^{c_1} f_{c_1 \ldots c_p} := \sum_{\sigma \text{ cyclic}} \mathrm{sign}(\sigma) \{\phi_{\sigma(a)}, f_{\sigma(b)\sigma(c_2)\ldots\sigma(c_p)}\}_{\mathrm{PB}} \tag{29}$$

where the σ is a cylic permutation of the letters $a, b, c_2, ..., c_p$, i.e., $\sigma \in S_{p+1}$. Then by construction

$$\hat{A} \;:\; \mathcal{C}^p \to \mathcal{C}^{p+1} \tag{30}$$

$$\hat{A}^2 \;=\; 0 \tag{31}$$

Hence we have a cohomology theory

$$\hat{Z}^p \;:=\; \mathrm{Ker}\hat{A}\Big|_{\mathcal{C}^p} \tag{32}$$

$$\hat{B}^p \;:=\; \mathrm{Im}\hat{A}\Big|_{\mathcal{C}^{p-1}} = \hat{A}\mathcal{C}^{p-1} \tag{33}$$

$$\hat{H}^p \;:=\; \hat{Z}^p / \hat{B}^p \tag{34}$$

By construction $\hat{H}^1$ measures the obstruction to invertibility of $\hat{A}$ as an operator on functions, and hence to the uniqueness of $\Phi_c^{(1)}$. The cohomology cannot be entirely trivial since

$$\hat{A}_{ab}{}^c \phi_c = 2 c_{ab}{}^c \phi_c \tag{35}$$

i.e., the constraints themselves are a kind of eigenvector system for $\hat{A}$.
Now, the standard complex computing the cohomology of a Lie algebra is the Koszul complex which is precisely our $\mathcal{C}^p$ complex, [5]. The derivative operator, however, is given by

$$sf(x_1, ..., x_{p+1}) = \sum_{q<r} (-1)^{q+r} f(x_1, ..., \hat{x_q}, ..., \hat{x}_r, ..., x_{p+1}, [x_q, x_r]) \tag{36}$$

where a hat over an x denotes that it is to be omitted and where $[\cdot, \cdot]$ denotes the appropriate Lie bracket (here the Poisson bracket). But the Poisson bracket is also a derivation, i.e., satisfies the Leibnitz rule

$$\{x, yz\}_{PB} = \{x, y\}_{\mathrm{PB}} z + y\{x, z\}_{\mathrm{PB}} \tag{37}$$

This is in fact one of the defining relations of a Poisson algebra, namely that it is a Lie algebra whose Lie bracket also satisfies the Leibnitz rule. From this one immediatly sees that $\hat{A} = s$ (possibly up to a sign), since

$$\begin{aligned} \{\phi, f^{1...p} x_1 ... x_p\}_{\mathrm{PB}} &= f^{1...p}\{\phi, x_1 ... x_p\}_{\mathrm{PB}} \\ &= f^{1...p} \sum_q \{\phi, x_q\}_{\mathrm{PB}} x_1 ... \hat{x}_q ... x_p \end{aligned}$$

Consequently $\hat{H}^*$ is nothing but the usual Lie algebra cohomology.
In a somewhat compact, symbolic notation the condition (43) then implies that $c\Phi$ and $\mathcal{J}$ differ by an exact term. Explicitly

$$c^c_{ab} \Phi^{(1)}_c = \{\phi_a, \phi_b\}_{\mathrm{PB}} + \text{exact form} \tag{38}$$

etc. For Lie algebras with vanishing cohomology we then see that the higher order terms $\Phi^{(n)}_a$ are completely determined by the ω_k brackets on the lower order terms. This ensures existence of such quantum constraints. It furthermore shows that equivalent quantum constraints only differ by exact terms, i.e., two set of quantum constraints are equivalent (give the same Moyal algebra) if and only if they are cohomologous (i.e., that they at each power of $\hbar$ only differ by an exact form).
A closely related cohomology generator will appear again when we deal with second class constraints.
This procedure of simply replacing Poisson brackets by Moyal ones has been used by Plebański and coworkers in the study of self-dual gravity, large N limit of $SU(N)$ theories and Nahm equations, [22]. The only new thing here is the level of generality – instead of considering a particular theory with constraints, we here deal with a generic situation.

2.1 Some Particular Solutions

It is quite natural to assume $\Phi_a \approx \phi_a$ (where $\approx$ denotes weak equivalence in the sense of Dirac), or more concretely that Φ_a only depends on (q,p) through the classical constraints ϕ_a. Assuming this, the first equation in the recursive scheme reads

$$c^c_{ab}\Phi^{(1)}_c(\phi) = c^d_{ac}\phi_d \frac{\partial\Phi^{(1)}_b}{\partial\phi_c} - c^d_{bc}\phi_d \frac{\partial\Phi^{(1)}_a}{\partial\phi_c} \tag{39}$$

a solution of which is $\Phi^{(1)}_c = \alpha^{ab}_c\phi_a\phi_b$ with α^{ab}_c a constant satisfying

$$\alpha^{ab}_c = \alpha^{ba}_c \tag{40}$$
$$2\alpha^{ef}_b c^g_{ae} - 2\alpha^{ef}_a c^g_{be} = c^c_{ab}\alpha^{gf}_c \tag{41}$$

This set of equations cannot always be guaranteed to have a non-trivial solution. Abelian constraint algebras are obviously not a problem, so let us consider the simplest non-Abelian algebra

$$\{\phi_1, \phi_2\}_{\mathrm{PB}} = \phi_2$$

in this case the structure coefficient is $c^2_{12} = 1$ and the equation for α^{ab}_c reduces to

$$(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\,\alpha^{gf}_2 = 2\left(\alpha^{2f}_b\delta_{a1} - \alpha^{1f}_b\delta_{a2} - \alpha^{2f}_a\delta_{b1} + \alpha^{1f}_a\delta_{b2}\right)\delta^{g2}$$

which has the solution

$$\begin{aligned} \alpha^{22}_f &= \text{arbitrary} \\ \alpha^{1f}_2 &= \alpha^{11}_1 = 0 \\ \alpha^{12}_1 &= -\frac{1}{2}\alpha^{22}_2 \end{aligned}$$

whence the quantum constraints read

$$\begin{aligned} \Phi_1 &= \phi_1 + \hbar\left(-\frac{1}{2}\phi_1 + \alpha^{22}_1\phi_2\right)\phi_2 \\ \Phi_2 &= \phi_2 + \hbar\alpha^{22}_2\phi^2_2 \end{aligned}$$

This then constitute an example with $\Phi_a \approx \phi_a$, where the quantum constraints depend on the phasespace variable only through the classical constraints. One should note that such a construction has found use in general

relativity, where a new Abelian constraint replacing the Hamiltonian constraint has been found, [14]. In that case, however, since the formal manipulations were carried out at a classical level, Planck's constant did not appear as a deformation parameter. Nevertheless, it is interesting to note the similarity. We will return to this new gravitational constraint algebra in a later section.
For the case of an su_2 algebra, $\{\phi_a,\phi_b\}_{\mathrm{PB}} = 2i\epsilon_{ab}{}^c\phi_c$, the conditions on the coefficients α_a^{bc} read

$$\begin{aligned}
\alpha_2^{2f}\delta_3^g - \alpha_2^{3f}\delta_2^g + \alpha_1^{1f}\delta_3^g - \alpha_1^{3f}\delta_1^g &= \frac{1}{2}i\alpha_3^{gf} \\
\alpha_3^{2f}\delta_3^g - \alpha_3^{3f}\delta_2^g - \alpha_1^{1f}\delta_2^g + \alpha_1^{2f}\delta_1^g &= -\frac{1}{2}i\alpha_2^{gf} \\
-\alpha_3^{1f}\delta_3^g + \alpha_3^{3f}\delta_3^g - \alpha_2^{1f}\delta_2^g + \alpha_2^{3f}\delta_1^g &= i\alpha_1^{gf}
\end{aligned}$$

which only has the trivial solution $\alpha_a^{bc}\equiv 0$, illustrating that not all algebras can be treated in this way. The example of an $O(N)$ non-linear σ-model belongs to this category as will be shown later.

2.2 The Physical States

Now, this treatment of constrained systems have dealt with merely the kinematics and not the dynamics proper, i.e., we have not studied the equations of motion. In the standard Dirac treatment of constrained Hamiltonian systems the first class constraints become operators upon quantisation, i.e., physical states have to satisfy

$$\hat{\phi}_a|\Psi\rangle = 0 \qquad \forall a \tag{42}$$

where $\hat{\phi}_a$ is some operator realisation of ϕ_a. How is this modified in a deformation quantisation?
A good starting point is to consider the deformation quantisation analogue of a state $|\Psi\rangle$, namely a *Wigner function* W_Ψ. We could then impose the constraints in the following algebraic way:

$$\Phi_a * W_\Psi = 0 \qquad \forall a$$

or more symmetrically (where $[\cdot,\cdot]_M^+$ denotes the "anti-Moyal bracket", $[f,g]_M^+ := f*g+g*f = 2f\cos(\frac{1}{2}\hbar\triangle)g$)

$$[\Phi_a, W_\Psi]_M^+ = 0 \qquad \forall a \tag{43}$$

Notice,

$$0 = [\Phi_a, W_\Psi]_M^+ = 2\Phi_a W_\Psi + ... \tag{44}$$

so to lowest order we have $W_\Psi \equiv 0$ away from the constraint surface. More precisely

$$\mathrm{supp} W_\Psi^{(0)} \subseteq \bigcap \mathrm{Ker}\phi_a \tag{45}$$

where $W_\Psi^{(0)}$ is the $\hbar^0$ component of W_Ψ.
The physical Hilbert space, $\mathcal{H}_{\text{phys}} \equiv \{|\Psi\rangle \mid \forall a : \hat{\phi}_a|\Psi\rangle = 0\}$, then gets replaced by the space

$$\mathcal{C}_{\text{phys}} := \{W \mid \forall a : [\Phi_a, W]_M^+ = 0\} \tag{46}$$

This also has the advantage of being analogous to the BRST-condition, $[\hat{\Omega}, A] = 0$ where $[\cdot,\cdot]$ is a graded commutator, $\hat{\Omega}$ is the BRST-charge, $\hat{\Omega} = \eta^a\hat{\phi}_a + ...$ and A is any observable (the observable corresponding to a state $|\Psi\rangle$ is of course the projection operator $A = |\Psi\rangle\langle\Psi|$). Thus we will be using (43) as defining the deformation quantisation analogue of the Dirac condition.
A few comments are in order. First, the replacement of Poisson brackets by Moyal ones implies that the corresponding "gauge" transformations acquire quantum modifications. If the classical constraints are ϕ_a, then they generate (infinitesimal) "gauge" transformations $\delta_\omega f := \{\omega^a\phi_a, f\}_{\text{PB}}$, the corresponding quantum version is

$$\delta_\omega F := [\omega^a\Phi_a, F]_M = i\hbar\{\omega^a\phi_a, F\}_{\text{PB}} + \text{other terms} \tag{47}$$

which a priori differs from the classical expression. The discrepancy between the classical and the quantum "gauge" transformations show up in higher order derivatives, which seems to suggest that the quantum transformations are "larger", i.e., slightly less local than their classical counterparts.
Another point to check is whether the space of physical quantities is invariant under such transformations. Consider thus an element A satisfying $[\Phi_a, A]_M^+ = 0,\ \forall a$, when then wants to prove that a "gauge" transformation does not take us away from this subspace, i.e., $\delta_\omega[\Phi_a, A]_M^+ = 0,\ \forall a$. We get

$$\begin{aligned}\delta_\omega([\Phi_a, A]_M^+) &= [\delta_\omega\Phi_a, A]_M^+ + [\Phi_a, \delta_\omega A]_M^+ \\ &= [\omega^b[\Phi_b, \Phi_c]_M, A]_M^+ + [\Phi_a, [\omega^b\Phi_b, A]_M]_M^+\end{aligned} \tag{48}$$

By noting that the physicality condition implies $\Phi_a * A = -A * \Phi_a,\ \forall a$, we can then rewrite this as

$$\delta_\omega([\Phi_a, A]_M^+) = \omega^b\left([[\Phi_b, \Phi_a]_M, A]_M^+ + [[\Phi_a, \Phi_b]_M, A]_M^+\right) = 0 \tag{49}$$

Hence the condition $0 = [\Phi_a, A]^+_M$ *is* a reasonable quantum analogue of $\phi_a = 0$, as we had anticipated, since it fulfills the only two requirements on can make *a priori*, namely that it has the correct classical limit (the correspondence principle) and that it is invariant under (quantum) gauge transformations. Notice that this even holds in the case where the structure coefficients depend upon the phasespace variables, as e.g. in gravity.
For completeness we recall how Wigner functions are formed. In standard quantum mechanics in d dimensions one defines [4]

$$W_\psi(q,p) = \int \psi^\dagger(x + \frac{1}{2}y) \otimes \psi(x - \frac{1}{2}y) e^{-iyp} \frac{dy}{(2\pi)^d} \tag{50}$$

where ψ is some wave function. This definition is appropriate in flat phasespaces and in the absence of external gauge fields. The case of gauge fields, however, can be treated by defining $\psi(x \pm \frac{1}{2}y)$ in a covariant manner using parallell transport, [6]. This can also be extended to quantum theory in curved spacetimes, [7]. More generally, one defines W_ψ by means of a *Wigner-Weyl-Moyal (WWM) map* as

$$W_\psi = \mathrm{Tr}(\Pi(q,p)|\psi\rangle\langle\psi|) \tag{51}$$

or for mixed states in terms of a density matrix ρ as $W_\rho = \mathrm{Tr}(\Pi(q,p)\rho(q,p))$, where Π encodes all of the deformation information, [8], i.e., an operator $\hat{A}$ gets mapped to a phasespace function A_W (its *Weyl symbol*)

$$A_W = \mathrm{Tr}(\Pi\hat{A}) \tag{52}$$

and where the deformed product is given by

$$A_W * B_W = \mathrm{Tr}(\Pi\hat{A}\hat{B}) = (\hat{A}\hat{B})_W \tag{53}$$

and so on. It has been proven that such a WWM-map Π exists for a very large range of algebraic structures, [8].
One of the main features of the WWM formalism is that states and observables are treated on an equal footing, in fact is given by a “projection operator” in $\mathcal{A}_\hbar$, i.e., an $*$-idempotent P, $P * P = P$. Thus $\mathcal{C}_{\text{phys}} \subseteq \mathcal{A}_\hbar$.
This can be clarified a bit by noting that the set of possible density matrices ρ, and hence of possible Wigner functions, is the (closed) convex hull of the set of $*$-idempotents (projections),

$$\mathcal{C} = \overline{\text{co}}\{P \in \mathcal{A}_\hbar \mid P * P = P\} \tag{54}$$

and consequently, $\mathcal{C}_{\text{phys}} \subseteq \mathcal{C}$.
As a final comment worth making, we could note the possibility of letting the

quantum constraints be the classical ones but with the usual product of phasespace variables replaced by the twisted one in a symmetric manner, i.e., replace qp by $\frac{1}{2}(q * p + p * q)$. But when these are Darboux coordinates, i.e., when they are canonically conjugate, then $\frac{1}{2}(q * p + p * q)$ and $qp = pq$ are identical, since $q*p = qp+i\hbar, p*q = qp-i\hbar$. It is only when we are working with a phasespace which is not covered by a global Darboux coordinate patch that this possibility have any relevance. Since the non-flatness of the phasespace can always be absorbed into an appropriate set of constraints on a larger, flat phasespace, this possibility is only of academic interest.

3 Overcoming the Various Problems

At the classical level, a number of problematic features can be present. The constraints may not all be first class or the constraints might not be in involution with the Hamiltonian, in both cases time evolution will take one away from the original constraint surface. We will show how to overcome these problems within the formalism of deformation quantisation. More precisely, we want to show that provided one is willing to allow negative powers of $\hbar$ (which is standard in the mathematical treatment of deformation quantisation), one can remove at the quantum level most if not all of the problems present at the classical level. It is this which is the main result of this paper.
Problems may also arrise after a naive attempt at quantisation, such problems will be referred to as *anomalies*, even though they may not be just the familiar cases of central extensions of the algebra (a.k.a. Schwinger terms in current algebra) but can encompass a wider range of problems. These too can be overcome at least in some cases as will be shown.

3.1 Second Class Constraints

In Dirac's terminology, second class constraints are constraints ψ_a which do not have weakly vanishing Poisson brackets, i.e.,

$$\{\psi_a, \psi_b\}_{\mathrm{PB}} = C_{ab} \not\approx 0 \qquad \det C \neq 0 \tag{55}$$

where being weakly zero means vanishing when the constraints have been taken into account (i.e., vanishing on the constraint surface). Dirac originally dealt with second class constraints by defining a new bracket, the *Dirac bracket*,

$$\{f, g\}_{\mathrm{DB}} = \{f, g\}_{\mathrm{PB}} - \{f, \psi_a\}_{\mathrm{PB}}(C^{-1})_{ab}\{g, \psi_b\}_{\mathrm{PB}} \tag{56}$$

which then have the wonderful property

$$\{\psi_a, \psi_b\}_{\mathrm{DB}} = 0 \tag{57}$$

i.e., the bracket of two second clas constraints become *strongly* zero.
There are a few problems with the Dirac bracket, however, most notably that it is usually very hard to compute explicitly. As far as deformation quantisation is concerned though, one can simply replace Poisson brackets by Dirac ones in the definition of the Moyal bracket to get a modified Moyal bracket. We will not pursue that line of inquiery any further here, since this approach will suffer even more from the calculational difficulties of the Dirac bracket. Instead we will ask two other questions: (1) is it possible to define "first class" quantum constraints $\Psi_a(\hbar)$ with $\Psi_a = ... + \psi_a + \hbar\Psi_a^{(1)} + ...$ (we will have to include negative powers of $\hbar$ in order to be able to cancel the C_{ab} from the zero'th order term), and (2) can the Dirac bracket itself be seen as a deformation of the Poisson bracket?
Let us first look for quantum constraints Ψ_a satisfying

$$[\Psi_a, \Psi_b]_M = i\hbar e_{ab}^c \Psi_c \tag{58}$$

where we explicitly want to be able to have $e_{ab}^c \equiv 0$ too. Since we now have to include negative powers of the deformation parameter $\hbar$, we write

$$\Psi_a = \sum_{n\in\mathbb{Z}} \hbar^n \Psi_a^{(n)} \qquad e_{ab}^c = \sum_{n=1}^{\infty} \hbar^n \overset{(n)}{e}{}_{ab}^{c} \tag{59}$$

and compute the Moyal bracket. The $\hbar^0$ part of this becomes

$$\sum_{n=1}^{\infty} \overset{(n)}{e}{}_{ab}^{c}\, \Psi_c^{(-n-1)} = \sum_{n\in\mathbb{Z}} \{\Psi_a^{(n)}, \Psi_b^{(-n)}\}_{\mathrm{PB}} + \sum_{k=1}^{\infty} \sum_{n\in\mathbb{Z}} \omega_k(\Psi_a^{(n)}, \Psi_b^{(-k-n)}) \tag{60}$$

which is then one of a family of equations the new constraints have to satisfy.
It will often be enough to assume $\Psi_c^{(n)} = 0, n = -2, -3, -4, ...$, i.e., that only one negative power of the deformation parameter occurs. In this instance the set of equations read

$$\begin{aligned} 0 &= \{\psi_a^{(-1)}, \psi_b^{(-1)}\}_{\mathrm{PB}}, & (61)\\ 0 &= \{\psi_a^{(-1)}, \psi_b\}_{\mathrm{PB}} + \{\psi_a, \psi_b^{(-1)}\}_{\mathrm{PB}}, & (62)\\ \overset{(1)}{e}{}_{ab}^{c}\, \psi_c^{(-1)} - C_{ab} &= \omega_3(\psi_a^{(-1)}, \psi_b^{(-1)}) + \{\psi_a^{(-1)}, \psi_b^{(1)}\}_{\mathrm{PB}} + \{\psi_a^{(1)}, \psi_b^{(-1)}\}_{\mathrm{PB}}, & (63) \end{aligned}$$

and so on. One would often like to take $e^c_{ab} \equiv 0$ but this might not always be possible, and not all choices of structure coefficients may be allowed. There will, however, be quite a lot of freedom in the particular choice of algebra in the general case.

As was the case for first class constraints, we can introduce a cohomology theory for this set of equations. Define

$$\hat{B}^{c_1}_{ab} f_{c_1 \ldots c_n} = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \{\psi^{(-1)}_{\sigma(a)}, f_{\sigma(b)\ldots\sigma(c_n)}\}_{\mathrm{PB}} \tag{64}$$

then $\hat{B}^2 = 0$. Notice that this is the same cohomology construction as for first class constraints, *except* that we now use the $\hbar^{-1}$ components and not the semiclassical ones for the definition of the differential. With this the conditions read

$$\hat{B}^c_{ab} \psi^{(-1)}_c = 0 \tag{65}$$

$$\hat{B}^c_{ab} \psi_c = 0 \tag{66}$$

$$\hat{B}^c_{ab} \psi^{(1)} = \overset{(1)}{e}{}^c_{ab}\, \psi^{(-1)}_c + C_{ab} - \omega_3(\psi^{(-1)}_a, \psi^{(-1)}_b) \tag{67}$$

So, contrary to the first class case, both the the $\hbar^{-1}$ and the $\hbar^0$ terms are closed. Let

$$\bar{Z}^p = \mathrm{Ker}\hat{B} \qquad \bar{B}^p = \mathrm{Im}\hat{B} \qquad \bar{H}^p = \bar{Z}^p/\bar{B}^p \tag{68}$$

then

$$\psi^{(-1)}_a, \psi_a \in \bar{Z}^1 \tag{69}$$

but

$$\psi^{(1)} \in \bar{Z}^1 + \omega_3(\bar{Z}^1, \bar{Z}^1) \tag{70}$$

which shows how the subsequent terms are build up from cohomological ingredient at lower levels, i.e., that the $\hbar$-term is a closed form plus the third order bracket, ω_3, of two closed forms. Hence, as for the case of first class constraints, the solutions are described by a cohomology. For second class constraints, however, the algebra of constraints is somewhat different. Instead of simply the algebra $\mathfrak{g} \subseteq \mathcal{A} = C^\infty(\Gamma)$ of classical constraints, ψ_a, we have an extended algebra $\tilde{\mathfrak{g}} \subseteq \mathcal{A} = C^\infty(\Gamma)$, which is an extension of $\mathfrak{g}$ by an Abelian algebra $\mathfrak{g}_0$ spanned by the $\hbar^{-1}$ components. The algebra $\mathfrak{g}_0$ defines a cohomology, $\bar{H}^p$, whose differential operator, $\hat{B}$, extends to all of the extended algebra $\tilde{\mathfrak{g}}$, making the situation somewhat more difficult. The extension $\tilde{\mathfrak{g}}$ need not be a central extension, since the condition $\hat{B}^c_{ab}\psi_c = 0$ does not imply $\{\psi_a, \psi^{(-1)}_b\}_{\mathrm{PB}} = 0$. It is clear, however, that central extensions would be a good starting point when one wants to construct $\tilde{\mathfrak{g}}$. This once more brings us back to the Lie algebra cohomology of $\mathfrak{g}$, since the second cohomology class thereof describes the various possible central extensions. We will not elaborate more on

the cohomological element here.

It is important to emphasise that the desire to remove second class constraints *forces* the pressence of negative powers of $\hbar$ upon us. But provided one is willing to pay this small price, one can treat second and first class constraints in the same manner *at the quantum level.* This of course implies that one has the redefine the classical limit slightly, a point to which we will return towards the end of this paper.

In the following subsection we will consider a particular example of a physical system with second class constraints and show how to use deformation quantisation explicitly. But first we will return to the Dirac bracket construction again.

The Dirac bracket, (56), can be seen as a deformation of the classical Poisson bracket in its own right. The deformation parameter is in this case a matrix valued function, namely C_{ab} itself. This is most easily seen in the following simple toy model where we have an algebra of first class constraints ϕ_a and a pair of second class ones, $\psi, \bar{\psi}$, satisfying the generic algebra

$$\begin{aligned} \{\phi_a, \phi_b\}_{\mathrm{PB}} &= c_{ab}^c \phi_c \\ \{\phi_a, \psi\}_{\mathrm{PB}} &= c_a^b \phi_b \\ \{\phi_a, \bar{\psi}\}_{\mathrm{PB}} &= \bar{c}_a^b \phi_b \\ \{\psi, \bar{\psi}\}_{\mathrm{PB}} &= k \end{aligned}$$

The Dirac bracket of the first class constraints then reads

$$\{\phi_a, \phi_b\}_{\mathrm{DB}} = c_{ab}^c \phi_c - k^{-1} c_a^c \bar{c}_b^d \phi_c \phi_d$$

One can either see this as defining a "quadratic algebra" or as a deformation of the Poisson bracket. This latter point is particularly clear when one considers two arbitrary functions f, g and compute their Dirac bracket

$$\{f, g\}_{\mathrm{DB}} = \{f, g\}_{\mathrm{PB}} - k^{-1} \{\psi, f\}_{\mathrm{PB}} \{\bar{\psi}, g\}_{\mathrm{PB}} := \{f, g\}_{\mathrm{PB}} - k^{-1} \omega(f, g)$$

Contrary to the Moyal bracket deformation of the classical Poisson brackets, the Dirac bracket, however, does not involve higher order derivatives of the observables. It is a deformation all the same; the limit $k \to \infty$ corresponding to the classical Poisson bracket. Contrary to the Moyal bracket, the higher oder brackets do not involve higher order derivatives. Deformations such that the n'th order bracket contains only n'th order derivatives is known as a deformation of *Vey type.* Hence, the Moyal bracket is of Vey type whereas the Dirac bracket is not.

The Dirac bracket amounts to replacing the Poisson bracket by a new first order bracket

$$\{f, g\}_2 = \alpha_{qq}(C) \frac{\partial f}{\partial q} \frac{\partial g}{\partial q} - \beta_{pq}(C) \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \beta_{qp}(C) \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} + \alpha_{pp}(C) \frac{\partial f}{\partial p} \frac{\partial g}{\partial p} \qquad (71)$$

where

$$\alpha_{pp}(C) = O(C) \qquad \alpha_{qq} = O(C) \tag{72}$$
$$\beta_{pq}(C) = -1 + O(C) \qquad \beta_{qp} = +1 + O(C) \tag{73}$$

where C is allowed to be a matrix valued function of the phasespace variables. We will use the name *secondary deformations* for such deformations. A priori, such secondary brackets will not define Lie algebras; one can only ensure this to $O(C)$. [4]

Before we move on to consider some examples and to study anomalies, it is worth recalling that other ways of removing second class constraints exist. These are inspired by the BRST approach to quantisation, [18]. One enlarges the phasespace by including Grassmann odd variables, $\eta, \mathcal{P}$, and then replaces the constraints ψ_i by $\Psi_i = \psi_i + \sum_{n=1} \psi_i^{(n)} \underbrace{\chi ... \chi}_{n}$, where χ can be either η or $\mathcal{P}$. The extra Grassmann variables (the ghosts) here correspond to our deformation parameter $\hbar$, and is just one indication of a connection between BRST and deformation quantisation. We will comment a bit more on this connection in the section on the underlying geometrical structure.

3.1.1 Example: $O(N)$ Non-Linear σ-Model

Consider an $O(N)$-valued field ϕ, and take the Lagrangean to be

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^A \partial^\mu \phi_A + \frac{1}{2}\lambda(\phi^A \phi_A - 1) \tag{74}$$

where μ is a (flat) spacetime index and A is the $O(N)$-index. In the Hamiltonian formalism of this $O(N)$ non-linear σ-model, we have the following constraints

$$\psi_1 = \phi_A \phi^A - 1 \tag{75}$$
$$\psi_2 = \pi^A \phi_A \tag{76}$$

where π_A is the canonically conjugate momentum of ϕ_A. The constraints are second class and satisfy

$$\{\psi_1(x), \psi_2(x')\}_{\rm PB} = 2\delta(x - x')\phi_A \phi^A \tag{77}$$

[4]This is somewhat similar to the situation in equivariant cohomology – if one attempts to use the secondary bracket $\{\cdot,\cdot\}_2$ to define a cohomology theory, i.e., a derivation s, one will get $s^2 = O(C)$ in the generic case, and one would not get a standard cohomology theory.

which is clearly non-zero on the constraint surface.
We now want to find quantum modified constraints $\Psi_i, i = 1, 2$ with vanishing Moyal brackets and with

$$\Psi_i = \hbar^{-1}\psi_i^{(-1)} + \psi_i \tag{78}$$

This leads to the following set of equations (suppressing the spacetime variables x, x' and the resulting Dirac δ-functions)

$$\begin{aligned} \{\psi_i^{(-1)}, \psi_j^{(-1)}\}_{\mathrm{PB}} &= 0 & (79)\\ \{\psi_i^{(-1)}, \psi_j\}_{\mathrm{PB}} + \{\psi_i, \psi_j^{(-1)}\}_{\mathrm{PB}} &= 0 & (80)\\ \omega_3(\psi_i^{(-1)}, \psi_j^{(-1)}) &= \{\psi_i, \psi_j\}_{\mathrm{PB}} = 2\epsilon_{ij}\phi_A\phi^A & (81)\\ \omega_n(\psi_i^{(-1)}, \psi_j^{(-1)}) &= 0 \qquad n \geq 5 & (82) \end{aligned}$$

We will make the following *Ansatz* for the quantum part of the constraints

$$\psi_1^{(-1)} = \psi_1 f(\phi) \qquad \psi_2^{(-1)} = \psi_2 g(\phi, \pi) \tag{83}$$

This yield the following set of coupled equations

$$\begin{aligned} 0 &= 2\phi^2 fg + \psi_1 g\phi \cdot \frac{\partial f}{\partial \phi} + \psi_1\psi_2 \frac{\partial f}{\partial \phi} \cdot \frac{\partial g}{\partial \pi} + 2f\psi_2\phi \cdot \frac{\partial g}{\partial \pi} & (84)\\ 0 &= 2\phi^2(f+g) + 2\psi_2\phi \cdot \frac{\partial g}{\partial \pi} + \psi_1\phi \cdot \frac{\partial f}{\partial \phi} & (85)\\ 0 &= \left(2\delta_{AB}\frac{\partial f}{\partial \phi_C} + 2\phi_A \frac{\partial^2 f}{\partial \phi_B \partial \phi_C} + \frac{1}{3}(\phi^2 - 1)\frac{\partial^3 f}{\partial \phi_A \partial \phi_B \partial \phi_C} + \text{cyclic}\right) \times \\ &\quad \left(\phi^A \frac{\partial^2 g}{\partial \pi^B \partial \pi^C} + \frac{1}{3}\phi \cdot \pi \frac{\partial^3 g}{\partial \pi^A \partial \pi^B \partial \pi^C} + \text{cyclic}\right) - 2\phi^2 & (86) \end{aligned}$$

where *cyclic* denotes a sum over cyclic permutations of A, B, C.
We can ensure $\omega_n(\Psi_1, \Psi_2) \equiv 0, n \geq 5$ if we take g to be only quadratic in the momenta, i.e.,

$$g(\phi, \pi) = \alpha_{AB}\pi^A\pi^B + \beta_A\pi^B + \gamma \tag{87}$$

where $\alpha_{AB}, \beta_A, \gamma$ can depend on ϕ. Notice, furthermore, that the ϕ-derivatives of g do not appear at all in this set of quations, hence there is a lot of freedom in choosing the ϕ-dependency of g. We will use this to assume α, β to be independent of ϕ.
By combining the first two equations we get

$$\begin{aligned} 2\phi^2 fg + 2f(\phi\cdot\pi)\phi^A(\alpha_{AB}\pi^B + \beta_A) &= -f(\phi^2 - 1)\phi \cdot \partial f & (88)\\ (g - f)\phi \cdot \partial f + (\phi \cdot \pi)(\alpha_{AB}\pi^B + \beta_A)\partial^A f &= 0 & (89) \end{aligned}$$

where we have written ∂f for $\frac{\partial f}{\partial \phi}$. Collecting powers of the momentum we get

$$\alpha_{AB}\pi^A\pi^B\phi^C\partial_C f = -\alpha_{BC}\pi^A\pi^B\phi^C\partial_A f \tag{90}$$
$$\beta_A\phi^B\pi_A\partial_B f = -\beta_A\phi^B\pi_B\partial_A f \tag{91}$$
$$(\gamma - f)\phi^A\partial_A f = 0 \tag{92}$$

To get an understanding of these equations we will consider first the case of $N = 1$. The third condition then implies $\gamma = f$. Using this we arrive at

$$2f' + 2\phi f'' + (\phi^2 - 1)f''' = 2\phi/\alpha \tag{93}$$

The form of this equation suggests an expansion on Hermite polynomials of $h = f'$. Thus, write

$$h = \sum_{n=0}^{\infty} c_n H_n(\phi) \tag{94}$$

Inserting this in (93) we get

$$\sum_{n=0}^{\infty}\left[c_n + 2\phi(n+1)c_{n+1} + 2(n+1)(n+2)(\phi^2-1)c_{n+2}\right]H_n = \frac{1}{2\alpha}H_1 \tag{95}$$

where we have used the explicit form for H_1 (i.e., $H_1(x) = 2x$) and the standard formula

$$\frac{d}{dx}H_n(x) = 2nH_{n-1}(x)$$

Now, multiplying by $H_1 e^{-\phi^2}$ and integrating over ϕ, we get from the orthonormality of the Hermite polynomials

$$2\sqrt{\pi}\left[c_1 + 12c_3 - 12c_3 + 480c_5 + 72c_3\right] = \frac{\sqrt{\pi}}{\alpha} \tag{96}$$

From this we get

$$h = c_1H_1 + c_3H_3 + c_5H_5 \tag{97}$$
$$= (2c_1 - 12c_3 + 120c_5)\phi + (8c_3 - 160c_5)\phi^3 + 32c_5\phi^5 \tag{98}$$

where the coefficients c_1, c_3, c_5 are restricted by (96). There are two undetermiend coefficients since the equation defining h is of second order. The function f is then the primitive of h, i.e.,

$$f(\phi) = c_0 + (c_1 - 6c_3 + 60c_5)\phi^2 + (2c_3 - 40c_5)\phi^4 + \frac{16}{3}c_5\phi^6 \tag{99}$$

where c_0 is an undetermined constant. Note, by the way, that, on the constraint surface, $f = const$. Hence our new quantum constraints are

$$\Psi_1 = \psi_1 \left(1 + \hbar^{-1} f(\phi)\right) \tag{100}$$
$$\Psi_2 = \psi_2 \left(1 + \hbar^{-1}(\alpha\pi^2 + \beta\pi + f(\phi))\right) \tag{101}$$

We have now constructed a set of quantum constraints Ψ_1, Ψ_2 satisfying an abelian algebra under the Moyal bracket,

$$[\Psi_1, \Psi_2]_M = 0 \tag{102}$$

and the second class constraints have consequently been "lifted" to an abelian, but deformed, algebra. This illustrates the general procedure of deformation quantisation of Hamiltonian systems with second class constraints. For the remaining part of this paper, we will implicitly assume that second class constraints have been dealt with in this manner, and will thus concentrate on first class constraints.
The case $N > 1$ is treated in an analogous manner, but the solution will here be in terms of many-variable Hermite polynomials, $H_{n_1,...,n_N}(\phi_1, ..., \phi_N)$ which satisfy similar recursion relations. Unfortunately, the explicit expressions quickly become highly complicated and messy and we will not give them here. It is, however, fairly straighforward to do so – it merely corresponds to finding solutions of the many-dimensional harmonic oscillator Schrödinegr equation. In a multi index notation, moreover, the equations essentialy reduce to the $N = 1$ case (with a few subtleties).

3.2 Constraints not in Involution with the Hamiltonian

Another problem which may occur is the failure of the constraints to be in involution with the Hamiltonian, i.e.,

$$\{h, \phi_a\}_{\mathrm{PB}} \neq V_a^b \phi_b \tag{103}$$

In this instance one cannot simply impose the constraints for some initial values of the phasespace variables, but have to impose them at each time step, since time evolution does not preserve the constraints.
Explicitly, we will assume the constraints to be first class but satisfying

$$\{h, \phi_a\}_{\mathrm{PB}} = V_a^b \phi_b + \chi_a \tag{104}$$

where $\chi_a \not\approx 0$. The Poisson bracket of χ_a with the constraints will be denoted by λ_{ab},

$$\{\phi_a, \chi_b\}_{\mathrm{PB}} = \lambda_{ab} \tag{105}$$

with $\lambda_{ab} \not\approx e^c_{ab}\chi_c$ for some function e^c_{ab}. This implies that χ_a cannot be seen as merely a new first class constraint.
The task is to find new quantum constraints and Hamiltonian such that

$$[H, \Phi_a]_M = i\hbar \tilde{V}^b_a \Phi_b \tag{106}$$

with $\tilde{V}^b_a = V^b_a + O(\hbar)$. We will again assume V^b_a to be independent of the phasespace coordinates.
Inserting an expansion in $\hbar$,

$$\begin{aligned} H &= \sum_n \hbar^n H_n \\ \Phi_a &= \sum_n \hbar^n \Phi^{(n)}_a \\ \tilde{V}^b_a &= \sum_n \hbar^n \overset{(n)}{V^b_a} \end{aligned}$$

we get for the $\hbar^0$ part

$$\sum_{k,l,n} \delta_{k+l,-2n} \omega_{2n+1}(H_k, \Phi^{(l)}_a) = \sum_n \overset{(n)}{V^b_a} \Phi^{(-n)}_b \tag{107}$$

which clearly shows that negative powers of $\hbar$ are needed – otherwise we would get the condition $V^b_a \phi_b + \chi_a = \overset{(0)}{V^b_a} \phi_b$ which is impossible. It does, however, make sense to assume that H and $\tilde{V}^b_a$ have only non-negative powers of the deformation parameter, whereas the only negative powers of $\hbar$ appear in Φ_a. We can even assume that only powers $\hbar^n, n \geq -1$ occur.
With this assumption, the condition coming from the $\hbar^{-1}$ part reads

$$\omega_1(h, \Phi^{(-1)}_a) = \overset{(0)}{V^b_a} \Phi^{(-1)}_b \tag{108}$$

i.e., the corresponding term in the quantum constraints *are* in involution with the Hamiltonian, with the structure coefficients being given by the involutive part of the classical relation, i.e., $\overset{(0)}{V^b_a} = V^b_a$.
The "classical" part, i.e., the $\hbar^0$ contribution, reads

$$V^b_a \phi_b + \chi_a + \omega_1(H_1, \Phi^{(-1)}_a) = \overset{(0)}{V^b_a} \phi_b + \overset{(1)}{V^b_a} \Phi^{(-1)}_b \tag{109}$$

which simplifies upon putting $\overset{(0)}{V}{}_a^b = V_a^b$ as mentioned above.
Since there are too many indeterminates in this problem (the H, Φ_a and $\tilde{V}_a^b$), we will make a further simplifying *Ansatz*, namely $H = h$, i.e., that the Hamiltonian receives no quantum corrections at this level – all quantum modifications will be put in the constraints and the structure coefficients. With this condition we arrive at

$$\overset{(1)}{V}{}_a^b \; \overset{(-1)}{\Phi}{}_b = \chi_a \tag{110}$$

Together with the involutive requirement, (108), this leads to a differential equation for $\overset{(1)}{V}{}_a^b$, namely

$$\{h, (\overset{(1)}{V}{}^{-1})_a^b \chi_b\}_{\mathrm{PB}} = V_a^b (\overset{(1)}{V}{}^{-1})_b^c \chi_c \tag{111}$$

Letting $v_a^b = \overset{(1)}{V}{}_a^b$ and using a matrix notation, this can be written as

$$\{h, v^{-1}\}_{\mathrm{PB}} \chi + v^{-1} \{h, \chi_b\}_{\mathrm{PB}} = V v^{-1} \chi \tag{112}$$

which we can also write as

$$v V v^{-1} \chi - \dot{\chi} = v \{h, v^{-1}\}_{\mathrm{PB}} \chi = -\{h, v\}_{\mathrm{PB}} v^{-1} \chi \equiv -\dot{v} v^{-1} \chi \tag{113}$$

In the simple case where we have only one constraint this equation becomes

$$V - \frac{d}{dt} \ln \chi = -\frac{d}{dt} \ln v$$

which has the solution

$$v(t) = c_0 e^{-Vt} \chi$$

where we have used that V is assumed independent of the phasespace coordinates in order to carry out the time integration.
Hence constraints which at the classical level aren't in involution with the Hamiltonian can, by deformation quantisation, be turned into quantum constraints which are. The price being that the constraints have a singular classical limit $\hbar \to 0$. This price turns out to be the standard fare when dealing with classical constraints with problems, or when one wants to eliminate problems arrising from a naive quantisation.

3.3 Anomalies and their Lifting

Upon deformation, the constraint algebra can be modified in a number of different ways. We will refer to such modification in general as *anomalies*. We have a natural hierarchy of such anomalies:

- Anomaly of zero'th order: $[\phi_a, \phi_b]_M \neq i\hbar\{\phi_a, \phi_b\}_{\text{PB}} = i\hbar c^c_{ab}\phi_c$.
- Anomaly of first order: $[\Phi_a, \Phi_b]_M \neq i\hbar c^c_{ab}\Phi_c$

The anomalies of zero'th order will in general be removable by deforming the constraints. One particularly important situation is when

$$\omega_n(\phi_a, \phi_b) = k^{(n)}_{ab} = const. \tag{114}$$

the Moyal bracket of two undeformed constraints is then

$$[\phi_a, \phi_b]_M = i\hbar c^c_{ab}\phi_c + k_{ab} \tag{115}$$

with $k_{ab} = \sum_n k^{(n)}_{ab}\hbar^{2n+1}$. Hence we have a central extension of the original constraint algebra. This is the way an anomaly very often shows up. We want to find deformed constraints, i.e., quantum constraints, Φ_a such that the anomaly is "lifted", $[\Phi_a, \Phi_b]_M = i\hbar\tilde{c}^c_{ab}\Phi_c$ – where we allow for a change in algebra for $\hbar \neq 0$. One should note that this procedure is analogous to the way we "lifted" second class constraints and the way in which we handled constraints not in involution with the Hamiltonian; the procedure thus shows the great flexibility and power of deformation quantisation.

It will in general be sufficient to assume $\overset{(n)}{c}{}^c_{ab} = 0$ for $n < 0$ and $\Phi^{(-k)}_a = 0$ for $k \geq 2$. Furthermore, suppose $k^{(n)}_{ab}$ vanishes after a certain stage N, it is then natural to choose $\overset{(n)}{c}{}^c_{ab} = \Phi^{(n)}_a = 0$ for $n \geq N$ too. Moreover, this will be the typical situation as the constraints usually are polynomial and hence have vanishing n-order brackets for n sufficiently large.
The general conditions become

$$\overset{(0)}{c}{}^c_{ab}\,\Phi^{(-1)}_c = \omega_1(\phi_a, \Phi^{(-1)}_b) + \omega_1(\Phi^{(-1)}_a, \phi_b) \tag{116}$$

$$\overset{(0)}{c}{}^c_{ab}\,\phi_c + \overset{(1)}{c}{}^c_{ab}\,\Phi^{(-1)}_c = \omega_1(\Phi^{(-1)}_a, \Phi^{(1)}_b) + \omega_1(\Phi^{(1)}_a, \Phi^{(-1)}_b) + c^c_{ab}\phi_c \tag{117}$$

$$\begin{aligned}\overset{(0)}{c}{}^c_{ab}\,\Phi^{(1)}_c + \overset{(1)}{c}{}^c_{ab}\,\phi_c + \overset{(2)}{c}{}^c_{ab}\,\Phi^{(-1)}_c = {} & \omega_1(\phi_a, \Phi^{(1)}_b) + \omega_1(\Phi^{(1)}_a, \phi_b) + \omega_1(\Phi^{(2)}, \Phi^{(-1)}_b) + \\ & \omega_1(\Phi^{(-1)}_a, \Phi^{(2)}_b) + \omega_3(\phi_a, \Phi^{(-1)}_b) + \omega_3(\Phi^{(-1)}_a, \phi_b)\end{aligned} \tag{118}$$

$$\begin{aligned}\overset{(0)}{c}{}^c_{ab}\,\Phi^{(2)}_c + \overset{(1)}{c}{}^c_{ab}\,\Phi^{(1)}_c + \overset{(2)}{c}{}^c_{ab}\,\phi_c + \overset{(3)}{c}{}^c_{ab}\,\Phi^{(-1)}_c = {} & \omega_1(\Phi^{(2)}_a, \phi_b) + \omega_1(\phi_a, \Phi^{(2)}_b) + \omega_1(\Phi^{(1)}_a, \Phi^{(1)}_b) + \\ & \omega_1(\Phi^{(-1)}_a, \Phi^{(3)}_b) + \omega_1(\Phi^{(3)}_a, \Phi^{(-1)}_b) + k^{(3)}_{ab}\end{aligned} \tag{119}$$

and so on.
We quickly recognize a cohomological element to these equations. In terms of the

aforementioned cohomology operators $\hat{A}^c_{ab}, \hat{B}^c_{ab}$, we can write the set of equations on the form

$$\begin{aligned}
\overset{(0)}{c}{}^c{}_{ab}\, \Phi^{(-1)}_c &= \hat{A}^c_{ab}\Phi^{(-1)}_c = \hat{B}^c_{ab}\phi_c && (120)\\
(\overset{(0)}{c}{}^c{}_{ab} - c^c_{ab})\phi_c + \overset{(1)}{c}{}^c{}_{ab}\, \Phi^{(-1)}_c &= \hat{B}^c_{ab}\Phi^{(1)}_c && (121)\\
\overset{(0)}{c}{}^c{}_{ab}\, \Phi^{(1)}_c + \overset{(1)}{c}{}^c{}_{ab}\, \phi_c + \overset{(2)}{c}{}^c{}_{ab}\, \Phi^{(-1)}_c &= \hat{A}^c_{ab}\Phi^{(1)}_c + \hat{B}^c_{ab}\Phi^{(2)}_c + \omega_3(\phi_a, \Phi^{(-1)}_b) + \omega_3(\Phi^{(-1)}_a, \phi_b) && (122)
\end{aligned}$$

etc., stating that the anomaly prevents $\Phi^{(-1)}_a$ from being $\hat{A}$-closed, or equivalently, ϕ_a from being $\hat{B}$-closed. In this way, an anomaly is recognized as an obstruction, and hence as corresponding to a non-trivial cohomology class.

Let us again for simplicity restrict ourselves to the simplest possible non-Abelian algebra $c^1_{12} = 1,\ a, b, c = 1, 2$. We will then write simply k for k_{12}. We can furthermore assume $\omega_{2n+1}(\phi_a, \phi_b) = 0$ for $n \geq 2$, i.e., $N = 2$ (a general value for N can be treated similarly but the notation quickly becomes cluttered). This implies we can take $\overset{(n)}{c^c_{ab}} = \Phi^{(n)}_a = 0, n \geq 2$ too. Moreover, it is natural in any case to pick $\overset{(0)}{c^c_{ab}} = c^c_{ab}$. The full set of conditions read

$$\begin{aligned}
\Phi^{(-1)}_2 &= \omega_1(\Phi^{(-1)}_1, \phi_2) + \omega_1(\phi_1, \Phi^{(-2)}_2) && (123)\\
\overset{(1)}{c^c_{12}}\, \Phi^{(-1)}_c &= \omega_1(\Phi^{(-1)}_1, \Phi^{(1)}_2) + \omega_1(\Phi^{(1)}_1, \Phi^{(-1)}_2) && (124)\\
\Phi^{(1)}_2 + \overset{(1)}{c^c_{12}}\, \phi_c &= \omega_1(\phi_1, \Phi^{(1)}_2) + \omega_1(\Phi^{(1)}_1, \phi_2) + \omega_3(\phi_1, \Phi^{(-1)}_2) + \\
&\quad\ \omega_3(\Phi^{(-1)}_1, \phi_2) && (125)\\
\overset{(1)}{c^c_{12}}\, \Phi^{(1)}_c &= \omega_1(\Phi^{(1)}_a, \Phi^{(1)}_2) + k && (126)\\
\omega_3(\phi_1, \Phi^{(1)}_2) + \omega_3(\Phi^{(1)}_a, \phi_2) &= 0 && (127)
\end{aligned}$$

It turns out, that a solution can be found if we make the *Ansätze*

$$\Phi^{(1)}_a = \alpha^b_a\phi_b + \beta_a \qquad \Phi^{(-1)}_a = \gamma^b_a\phi_b \qquad (128)$$

where α, β, γ are constants. The appearance of a β is needed because of the fourth relation above would otherwise yield $k \propto \phi$ which is explicitly assumed not to be the case. Furthermore, the first relation shows that $\Phi^{(-1)}_a$ can only be proportional to ϕ_a and not be linearly dependent upon it.

By inserting the *Ansätze* in the relations above, one gets

$$\gamma^1_2 = \gamma^1_1 = 0$$

$$
\begin{aligned}
\overset{(1)}{c}{}^{1}_{12}\ \gamma^2_1 &= \alpha^1_2\gamma^2_1 + \alpha^1_1\gamma^2_2 \\
\mathrm{Tr}\alpha &= \alpha^2_2 + \overset{(1)}{c}{}^{2}_{12} \\
\alpha^1_2 + \overset{(1)}{c}{}^{1}_{12} &= 0 \\
\beta_2 &= \gamma^2_2 k \\
\det\alpha &= \overset{(1)}{c}{}^{c}_{12}\ \alpha^2_c \\
\overset{(1)}{c}{}^{c}_{12}\ \alpha^1_c &= 0 \\
\overset{(1)}{c}{}^{c}_{12}\ \beta_c &= k
\end{aligned}
$$

From which we get the structure coefficients

$$
\overset{(1)}{c}{}^{c}_{12} = \begin{cases} -\alpha^1_2 & c=1 \\ \alpha^1_1 & c=2 \end{cases} \tag{129}
$$

and the linear part, β, to be

$$
\beta_1 = \left(\gamma^2_2 - \frac{1}{\alpha^1_1}\right)k \qquad \beta_2 = \gamma^2_2 k \tag{130}
$$

Which shows, that it is the central extension, k, which necessitates the linear part β. Finally we get for the matrices α, γ that they must be of the form

$$
\alpha = \begin{pmatrix} \alpha_1 & -\alpha_1 \\ \alpha_2 & -\alpha_1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 0 & \gamma_1 \\ 0 & \gamma_2 \end{pmatrix} \tag{131}
$$

subject to

$$
-2\alpha_2\gamma_1 = \alpha_1\gamma_2 \tag{132}
$$

but otherwise they can be chosen at random (but non-zero, otherwise $\gamma \equiv 0$ or $\alpha \equiv 0$ which is clearly not what we want).
Explicitly, the new quantum constraints read

$$
\begin{aligned}
\Phi_1 &= \hbar^{-1}\gamma_1\phi_2 + \phi_1 + \hbar\alpha_1(\phi_1 - \phi_2) + \hbar(\gamma_2 - \alpha_1^{-1})k && (133) \\
\Phi_2 &= \hbar^{-1}\gamma_2\phi_2 + \phi_2 + \hbar(\alpha_2\phi_1 - \alpha_1\phi_2 + \gamma_2 k) && (134)
\end{aligned}
$$

and the Moyal algebra is

$$
[\Phi_1, \Phi_2]_M = (1 - \alpha_2\hbar)\Phi_1 + \alpha_1\hbar\Phi_2 \tag{135}
$$

which is still isomorphich to the classical, undeformed Poisson algebra, since that was the unique non-Abelian algebra with two generators. In general the Moyal and the Poisson algebra needn't be isomorphic, however, the latter only being a limit (or contraction), $\hbar \to 0$, of the former.
The important thing to learn from this example is that the quantum constraints acquire quantum modifications of negative order in $\hbar$ as well as of positive orders, and that, furthermore, the structure coefficients too get quantum modified, but only with positive powers of $\hbar$. Note, by the way, that it would be inconsistent to choose $\tilde{c}^c_{ab} = c^c_{ab}$, hence anomalies of the zero'th order can be lifted, provided one is willing to modify the structure coefficients.
In the case where k_{ab} is not a constant, we will have to let α, β, γ too depend on the phasespace variables. In this way, the conditions on these coefficient become differential equations, just like they did for the case of constraints not in involution with the Hamiltonian, and we will not comment further on that here.
Anomalies of first order can result from a bad choice of quantum constraints, but if that is not the case, they cannot in general be lifted. It is difficult to think of any examples of such systems; they would correspond to a set-up in which the only consistent quantisations were drastically different from their classical counterparts. If the Moyal algebra becomes simply a non-linear algebra, i.e., if for instance

$$[\Phi_a, \Phi_b]_M = i\hbar {c_{ab}}^c \Phi_c + i\hbar^2 {d_{ab}}^{cd} \Phi_c \Phi_d$$

then we can lift this anomaly by extending the constraint algebra to include $\Phi_{ab} := \hbar \Phi_a \Phi_b$. Typically, this would lead to an infinite dimensional constraint algebra.
Hamachi, [20], has recently considered another kind of anomaly "smoothening". Again, it turns out that anomalous contributions can be removed at the Moyal algebra level. However, Hamachi restricts himself to systems with constraints only depending on momentum. The set-up presented here is much more general, although our results are not as rigorous as his. Just like in our case, the anomaly reappears as a troublesome $\hbar \to 0$ limit.

4 Examples: Yang-Mills and Gravity

Possibly the two most important kind of theories in modern theoretical physics are Yang-Mills theories and General Relativity together describing all known forces in the universe. It is consequently of interest to devote some time to the study of their deformation quantisation. Especially when one considers the important differences in the structure of their respective constraint algebras. Where the structure coefficients of Yang-Mills theory are independent of the fields (thereby forming a true Lie algebra), the corresponding coefficients of the gravitational case

do depend on the fields (the metric to be precise). Furthermore, at least two set of constraint algebras exist for general relativity, the ADM ones and the Ashtekar variables, the latter formally resembling Yang-Mills theory quite a bit. Recently, a third algebra of constraints for gravity has been proposed, where the algebra is of the form $C^\infty(\Sigma) \ltimes \mathrm{Diff}(\Sigma)$ where Σ is the Cauchy hypersurface and $\ltimes$ denotes semidirect product. We will return to these after having dealt with Yang-Mills theory.

4.1 Yang-Mills Theory

For a Yang-Mills theory with some gauge algebra $\mathfrak{g}$, the constraints are

$$\mathcal{G}_a = D_i \pi^i_a \tag{136}$$

where π^i_a is the momentum conjugate to A^a_i, D_i denotes the gauge covariant derivative and i is a space index (which we will take to run from one to three) and a is a Lie algebra index. The constraint algebra is

$$\{\mathcal{G}_a(x), \mathcal{G}_b(x')\}_{\mathrm{PB}} = c^c_{ab}\delta(x-x')\mathcal{G}_c(x) \tag{137}$$

where c^c_{ab} are the structure coefficients of $\mathfrak{g}$.
Since the constraints are quadratic in the variables (they have the form $\mathcal{G} \sim \partial\pi + \pi A$), we can take $\mathcal{G}_a$ to be the quantum constraints too, i.e.,

$$[\mathcal{G}_a(x), \mathcal{G}_b(x')]_M = i\hbar c^c_{ab}\delta(x-x')\mathcal{G}_c(x) \tag{138}$$

The last bracket is the one with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\pi^i_a \pi^a_i - \frac{1}{2}F^a_{ij}F^{ij}_a \tag{139}$$

Although $\mathcal{H}$ is fourth order in the fields (it has an A^4 term) the constraints are merely quadratic and hence $\omega_n(\mathcal{H}, \mathcal{G}_a) = 0$ for $n \geq 1$. Thus we can take $\mathcal{H}$ to be also the quantum Hamiltonian, and we get no anomalies (i.e., no anomalies from the deformation quantisation procedure, that is, other types of anomalies, especially global ones, are still possible).
The constraint equations become finite order functional differential equations

$$0 = [\mathcal{G}_a, W]^+_M = 2(D_i\pi^i_a)W + \frac{1}{4}i\hbar^2\delta^j_k c^c_{ab}\frac{\delta^2 W}{\delta A^c_j \delta\pi^k_b} \tag{140}$$

where we have used $[f,g]^+_M = 2f\cos(\frac{1}{2}\Delta)g$,[10]. We have also suppressed gauge indices on the Wigner function (it has two – it taking values in $\mathfrak{g}\otimes\mathfrak{g}^*$, [6, 7]). Formally, a solution can be found of the form

$$W[A,\pi] = e^{\mathrm{Tr}\int F_{ij}g^{ij}dx} \tag{141}$$

where $g^{ij} = g^{ij}(\pi)$ is a quadratic function with values in the Lie algebra, $g_a^{ij} = d_a^{\ bc}\pi_b^i\pi_c^j$ where $d_a^{\ bc}$ is antisymmetric in bc and satisfies

$$c^c_{\ ab}d_c^{\ ba} = 8i\hbar^{-2} \tag{142}$$

where indices are raised and lowered by the Kronecker delta. Since we already have one invariant antisymmetric tensor on $\mathfrak{g}$, namely the structure coefficient, $c^c_{\ ab}$, it is natural to put

$$d_c^{\ ab} = \alpha c_c^{\ ab} \tag{143}$$

which leads to

$$\alpha = -\frac{8i}{\hbar^2\kappa} \tag{144}$$

where $\kappa = \delta^{ab}\kappa_{ab}$ is the trace of the Cartan-Killing metric, $\kappa_{ab} = c^c_{\ ad}c^d_{\ bc}$. For $\mathfrak{g} = su_n$ for instance, we have $\kappa = -2$, leading to

$$W[A,\pi] = \exp\left(-4i\hbar^{-2}\int c_a^{\ bc}F_{ij}^a\pi_b^i\pi_c^j dx\right)$$

For su_2, teh Wigner function then involves the dual of the field strength tensor, whereas in other cases it involves some kind of generalised dual. In any case, the Wigner function is a kind of "bi-Gaussian", i.e., a Gaussian in either A or π with the coeffients depending on the conjugate variable.
It is interesting to note that the present solution cannot be the Wigner function of a pure state. This is seen as follows. First, the Wigner function has the form

$$W(q,p) = e^{(aq+bq^2)p^2}$$

hence the Fourier transform $p \rightarrow y$ looks like

$$W(q,y) = \sqrt{\frac{\pi}{aq+bq^2}}e^{-\frac{1}{4}\frac{y^2}{aq+bq^2}}$$

By the definition of the Wigner function of a pure state described by a wave function ψ, we have

$$W(q,0) = |\psi(q)|^2$$

hence

$$\psi(q) = \left(\frac{\pi}{aq+bq^2}\right)^{1/4} e^{i\theta(q)}$$

where θ is purely real. But, on the other hand,

$$W(q,y) = \sqrt{\frac{\pi}{aq+bq^2}}e^{-\frac{1}{4}\frac{y^2}{aq+bq^2}} \propto \bar{\psi}(q+\frac{1}{2}y)\psi(q-\frac{1}{2}y)$$

is only possible if θ has an imaginary part.
The equation for the Wigner function of a Yang-Mills gauge field, found by Elze, Gyulassy and Vasak, [6], uses second quantised quantum mechanics and hence operate on the mechanical phasespace (q, p) and not on the field theoretic one (A, π). Consequently, they get an infinite order partial differential equation whereas we get a finite order functional differential one. We do also, however, get an infinite order equation, since for Yang-Mills theory not all equations of motion are constraints, only part of them (corresponding to Gauss' law). The remaining equations must be found by the same techniques as used by Elze, Gyulassy and Vasak, but this time in a purely Hamiltonian framework.
One should note that for abelian theories, the second term in (140) vanishes and we get the classical requirement $\mathcal{G}_a \equiv 0$ (since $W \neq 0$ in general – it is, a priori, possible to have $\mathcal{G}_a \neq 0$ at a point (q, p) provided $W(q, p) = 0$ of course, hence the zeroes of the Wigner function can correspond to points in phasespace at which the classical constraints are violated, by continuity of W this can only happen in a discrete number of points or in a region disconnected from the rest of phasespace). Thus, it is the non-abelianness of the gauged algebra, $\mathfrak{g}$, which leads to quantum deformations of the conditions of physical degrees of freedom.
This corresponds to what was proven in [8] where it was shown that a generalised WWM-formalism exists for a large range of algebraic structures, the "deformation" of the resulting "classical" phasespace (in particular its curvature), being due to the non-commutativity of the generators of the algebra.
It is not surprising that Yang-Mills theory is anomaly free at this level, since anomalies tend to appear through the Dirac operator, and hence through the matter fields, [12, 13]. We have only considered Yang-Mills theory *in vacuo* at this stage. The usual anomalies (chiral, parity) should then reappear only when one computes the Moyal bracket for the fermion currents. It is already known that the WWM symbol of an operator is related to the index of it, and that the Atiyah-Singer index theorem normally used to express anomalies is intimately related to the entire WWM-scheme, [24]. One should also take notice of the fact that the above discussion doesn't take global anomalies into account, such problems are beyond the scope of the present paper.

4.2 Gravitation

For gravity, we will first consider the ADM constraint algebra, then the Ashtekar variables and finally make some brief comments on the newly proposed "Kuchař algebra.", [14, 15]

The ADM constraints, [11], are

$$\mathcal{H}_\perp(x) = g^{-1/2}(\frac{1}{2}\pi^2 - \pi^i_j\pi^j_i) + \sqrt{g}R = G_{ijkl}\pi^{ij}\pi^{kl} + \sqrt{g}R \tag{145}$$

$$\mathcal{H}_i(x) = -2D_j\pi^j_i \tag{146}$$

with g_{ij} the 3-metric, π^{ij} its conjugate momentum,

$$\{g_{ij}(x), \pi^{kl}(x')\}_{\text{PB}} = \frac{1}{2}(\delta^k_i\delta^l_j + \delta^k_j\delta^l_i)\delta(x,x'), \tag{147}$$

R the curvature scalar of g_{ij} (i.e., the three dimensional one) and g the determinant of the 3-metric. The first constraint is known as the Hamiltonian one, and the last, the $\mathcal{H}_i$, as the diffeomorphism one. The algebra is

$$\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\}_{\text{PB}} = (g^{ij}(x)\mathcal{H}_j(x) + g^{ij}(x')\mathcal{H}_j(x'))\delta_{,i}(x,x') \tag{148}$$

$$\{\mathcal{H}_\perp(x), \mathcal{H}_i(x')\}_{\text{PB}} = \mathcal{H}_\perp(x)\delta_{,i}(x,x') \tag{149}$$

$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\}_{\text{PB}} = \mathcal{H}_i(x')\delta_{,j}(x,x') + \mathcal{H}_j(x)\delta_{,i}(x,x') \tag{150}$$

where the subscript, $\delta_{,i}$, on the delta function denotes partial derivative with respect to x^i. The convention is the standard one in which $\delta(x,x')$ is a scalar in the first argument and a density in the second (the curved spacetime Dirac δ has a $g^{-1/2}$ in it).

The algebra of the diffeomorphism constraint will not be deformed as their form is $\mathcal{H}_i \sim \partial\pi + g^2\pi$ and thus has $\omega_3 \equiv 0$. The Hamiltonian constraint, however, has as well a $g\pi^2$ as a $g^2\pi$ term, and will consequently not have vanishing ω_3. We should thus expect the algebraic relations involving $\mathcal{H}_\perp$ to receive $\hbar^3$ corrections (but no higher order corrections since no higher powers of π are present). This is precisely what we find. Moreover, the Christoffel symbols and the $\sqrt{g}$ contain, in a Taylor series, the metric to infinite order, whence we should expect infinite order equations to turn up at some stage.

A straightforward computation yields

$$[\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')]_M = i\hbar\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\}_{\text{PB}} + \hbar^3 k(x,x') \tag{151}$$

where

$$k(x,x') = -\frac{1}{8}i\left((\Xi^{ijkl}_{mnab}\pi^{ab})(x)\mathcal{G}^{mn}_{ijkl}(x') - (x \leftrightarrow x')\right) \tag{152}$$

with

$$\Xi^{ijkl}_{mnab} \equiv \frac{\delta^3\mathcal{H}_\perp}{\delta g_{ij}\delta g_{kl}\delta\pi^{mn}} \tag{153}$$

$$= G_{mnab}(g^{ik}g^{jl} - \frac{1}{2}g^{ij}g^{kl}) - \frac{1}{2}g^{ij}\left(g_{nb}\delta^k_m\delta^l_a + g_{ma}\delta^k_n\delta^l_b - \right.$$

$$g_{ab}\delta^k_m\delta^l_n - g_{mn}\delta^k_a\delta^l_b + g_{bn}\delta^l_m\delta^k_a + g_{am}\delta^l_n\delta^k_b\Big) \tag{154}$$

$$\mathcal{G}^{mn}_{ijkl} \equiv \frac{\delta^3\mathcal{H}_\perp}{\delta\pi^{ij}\delta\pi^{kl}\delta g_{mn}} \tag{155}$$

$$= -\frac{1}{2}g^{mn}G_{ijkl} + g^{-1/2}\left(g_{jl}(\delta^m_i\delta^n_k + \delta^m_k\delta^n_i) + g_{ik}(\delta^m_j\delta^n_l + \delta^m_l\delta^n_j) - g_{kl}\delta^m_i\delta^n_j - g_{ij}\delta^m_k\delta^n_l\right) \tag{156}$$

One should note that $[\mathcal{H}_\perp, k]_M \neq 0$ hence we get an anomaly which is not even a central extension of the original algebra. Explicitly

$$[\mathcal{H}_\perp, k]_M = i\hbar\{\mathcal{H}_\perp, k\}_{\mathrm{PB}} + i\hbar^3\frac{3}{4}\Xi^{ijkl}_{mnab}\pi^{ab}\frac{\delta^3 k}{\delta\pi^{ij}\delta g_{kl}\delta g_{mn}} \neq 0 \tag{157}$$

For the ADM constraints, the structure coefficients depend on the fields, consequently the anomaly too depends upon (g, π).
Similarly, the relation mixing $\mathcal{H}_\perp$ and $\mathcal{H}_i$ receives a $\hbar^3$ correction of the form

$$k_i(x, x') \equiv -\frac{1}{8}i\frac{\delta^3\mathcal{H}_\perp(x)}{\delta\pi^{jk}\delta\pi^{lm}\delta g_{ab}}\frac{\delta^3\mathcal{H}_i(x')}{\delta g_{jk}\delta g_{lm}\delta\pi^{ab}}$$

which one easily finds to be

$$k_i(x, x') = -\frac{1}{4}i\mathcal{G}^{pq}_{jklm}\Upsilon^{jklm}_{ipq} \tag{158}$$

with

$$\begin{aligned}
\frac{\delta^3\mathcal{H}_i(x)}{\delta g_{jk}(x')\delta g_{lm}(x'')\delta\pi^{ab}(y)} &= 2\Upsilon^{jklm}_{iab}(x, x', x'')\delta(x, y) \\
&= \delta^c_{(a}\delta^n_{b)}\delta(x, y)\Big\{\delta^l_{(n}\delta^m_{i)}\delta(x, x'')\Big(-g^{rk}\Gamma^j_{rc}\delta(x, x') + \\
&\quad \frac{1}{2}g^{rs}\left(\delta^j_{(r}\delta^k_{s)}\partial_c + \delta^j_{(s}\delta^k_{c)}\partial_r - \delta^j_{(c}\delta^k_{r)}\partial_s\right)\delta(x, x')\Big) + \\
&\quad \delta^j_{(n}\delta^k_{i)}\delta(x, x')\Big(-g^{rm}\Gamma^l_{rc}\delta(x, x'') + \\
&\quad +\frac{1}{2}g^{rs}\left(\delta^l_{(r}\delta^m_{s)}\partial_c + \delta^l_{(s}\delta^m_{c)}\partial_r - \delta^l_{(c}\delta^m_{r)}\partial_s\right)\delta(x, x'')\Big) - \\
&\quad g_{ni}\delta(x, x'')\Big[g^{rl}g^{km}\Gamma^j_{rc} - g^{rk}g^{jm}\Gamma^l_{rc} + \\
&\quad \frac{1}{2}g^{rk}g^{js}\left(\delta^l_{(r}\delta^m_{s)}\partial_c + \delta^l_{(s}\delta^m_{c)}\partial_r - \delta^l_{(r}\delta^m_{c)}\partial_s\right) - \\
&\quad \frac{1}{2}g^{rl}g^{sm}\left(\delta^j_{(r}\delta^k_{s)}\partial_c + \delta^j_{(s}\delta^k_{c)}\partial_r - \delta^j_{(r}\delta^k_{c)}\partial_s\right)\Big]\delta(x, x')\Big\} + \\
&\quad (c \to r, n \to c, r \to n)
\end{aligned} \tag{159}$$

The spatial diffeomorphism subalgebra spanned by $\mathcal{H}_i$ does not receive any quantum corrections since the constraints are only linear in the momentum.
The set of physical states are defined as the functions W satisfying the two infinite order functional differential equations

$$\begin{aligned} 0 &= [\mathcal{H}_\perp, W]_M^+ && (160) \\ 0 &= [\mathcal{H}_i, W]_M^+ && (161) \end{aligned}$$

these are infinite order since the Christoffel symbols (and hence the covariant derivative and the curvature scalar) has an inverse metric in them, similarly the supermetric G_{ijkl} too has an inverse metric inside. Thus the constraints are not polynomial in the metric, but instead "meromorphic".
Written out more explicitly, the physicality conditions read

$$\begin{aligned} 0 &= 2\mathcal{H}_\perp W + \sum_{k=1}^{\infty}(-1)^k 2^{-2k-1}\hbar^{2k}\left(\frac{\delta^{2k}\mathcal{H}_\perp}{\delta g_{i_1j_1}...\delta g_{i_{2k-1}j_{2k-1}}\delta g_{mn}}\frac{\delta^{2k}W}{\delta\pi^{i_1j_1}...\delta\pi^{i_{2k-1}j_{2k-1}}\delta\pi^{mn}}-\right. \\ & 2k\frac{\delta^{2k}\mathcal{H}_\perp}{\delta g_{i_1j_1}...\delta g_{i_{2k-1}j_{2k-1}}\delta\pi^{mn}}\frac{\delta^{2k}W}{\delta\pi^{i_1j_1}...\delta\pi^{i_{2k-1}j_{2k-1}}\delta g_{mn}}+ \\ & \left. k(2k-1)\frac{\delta^{2k}\mathcal{H}_\perp}{\delta g_{i_1j_1}...\delta g_{i_{2k-2}j_{2k-2}}\delta\pi^{i_{2k-1}j_{2k-1}}\delta\pi^{mn}}\frac{\delta^{2k}W}{\delta\pi^{i_1j_1}...\delta\pi^{i_{2k-2}j_{2k-2}}\delta g_{i_{2k-1}j_{2k-1}}\delta g_{mn}}\right) \end{aligned} \tag{162}$$

$$\begin{aligned} 0 &= 2\mathcal{H}_i W + \sum_{k=1}^{\infty}(-1)^k 2^{-2k-1}\hbar^{2k}\left(\frac{\delta^{2k}\mathcal{H}_i}{\delta g_{i_1j_1}...\delta g_{i_{2k}j_{2k}}}\frac{\delta^{2k}W}{\delta\pi^{i_1j_1}...\delta\pi^{i_{2k}j_{2k}}}-\right. \\ & \left. 2k\frac{\delta^{2k}\mathcal{H}_i}{\delta g_{i_1j_1}...\delta g_{i_{2k-1}j_{2k-1}}\delta\pi^{i_{2k}j_{2k}}}\frac{\delta^{2k}W}{\delta\pi^{i_1j_1}...\delta\pi^{i_{2k-1}j_{2k-1}}\delta g_{i_{2k}j_{2k}}}\right) \end{aligned} \tag{163}$$

Since the constraints for gravity in the ADM formalism are non-polynomial the equations defining the physical state space become infinite order. If one assumes the Wigner function to be analytic in $\hbar$, one can Taylor expand it $W = \sum_{n=0}^{\infty}\hbar^n W_n$, and arrive at the following recursive formulas for the n'th order coefficients, W_n

$$0 = \mathcal{H}_\perp W_0 = \mathcal{H}_i W_0 \tag{164}$$

$$\begin{aligned} 0 &= 2\mathcal{H}_\perp W_N + \sum_{k=1}^{[N/2]}(-1)^k 2^{-2k-1}\left(\frac{\delta^{2k}\mathcal{H}_\perp}{\delta g_{i_1j_1}...\delta g_{i_{2k-1}j_{2k-1}}\delta\pi^{mn}}\frac{\delta^{2k}}{\delta\pi^{i_1j_1}...\delta\pi^{i_{2k-1}j_{2k-1}}\delta g_{mn}}-\right. \\ & \left. 2k\frac{\delta^{2k}\mathcal{H}_\perp}{\delta g_{i_1j_1}...\delta g_{i_{2k-2}j_{2k-2}}\delta\pi^{i_{2k-1}j_{2k-1}}\delta\pi^{mn}}\frac{\delta^{2k}}{\delta\pi^{i_1j_1}...\delta\pi^{i_{2k-2}j_{2k-2}}\delta g_{i_{2k-1}j_{2k-1}}\delta g_{mn}}\right)W_{N-2k} \end{aligned} \tag{165}$$

$$0 \quad = \quad 2\mathcal{H}_i W_N + \sum_{k=1}^{[N/2]} (-1)^k 2^{-2k-1} \frac{\delta^{2k}\mathcal{H}_i}{\delta g_{i_1 j_1} \ldots \delta g_{i_{2k-1} j_{2k-1}} \delta\pi^{mn}} \frac{\delta^{2k} W_{N-2k}}{\delta\pi^{i_1 j_1} \ldots \delta\pi^{i_{2k-1} j_{2k-1}} \delta g_{mn}} \tag{166}$$

where $k, N \geq 1$. These equations are not enough to completely specify the Wigner functions.

The conclusion so far is then that in the ADM formalism gravity is anomalous when one attempts a deformation quantisation. The question is, then, whether one can lift these anomalies or not.

Letting $\mathcal{H}_0 = \mathcal{H}_\perp$ and $\mu = (0, i)$ we then want quantum constraints H_μ satisfying

$$[H_\mu(x), H_\nu(x')]_M = c^\rho_{\mu\nu}(x, x') H_\rho(x) + d^\rho_{\mu\nu}(x, x') H_\rho(x') \tag{167}$$

where the structure coeffcients $c^\rho_{\mu\nu}, d^\rho_{\mu\nu}$ depend on the coordinates only through Dirac delta functions (and their derivatives) and directly through the phasespace variables. Because of the complicated nature of the constraints in the ADM formalism I have not been able to find a good set of quantum constraints.

We saw that the anomalous nature of the quantum deformed algebra of the contraints in the ADM formalism were due to the constraints being non-polynomial. It is therefore interesting ot consider another formulation, the Ashtekar variables [16], where the constraints *are* polynomials. In this formulation the canonical coordinates are a complex su_2-connection A^a_i and its momentum (a densitised *dreibein*) E^i_a, and the constraints are

$$\begin{aligned} \mathcal{H} &= F^a_{ij} E^i_b E^j_c \varepsilon^{bc}{}_a & (168) \\ \mathcal{G}_a &= D_i E^i_a & (169) \\ \mathcal{D}_i &= F^a_{ij} E^j_a & (170) \end{aligned}$$

It will turn out that in these variables the anomaly is much simpler, namely merely a central extension.

Since the Ashtekar variables bring out the analogy between general relativity and (complexified) Yang-Mills theory due to the isomorphism $so(3,1) \simeq su_2 \otimes \mathbb{C}$, we can use our knowledge of the deformation quantisation of Yang-Mills systems to see that only the following two brackets can receive any quantum corrections, and these only to lowest order

$$\big[\mathcal{H}(x), \mathcal{H}(x')\big]_M \quad = \quad i\hbar\{\mathcal{H}(x), \mathcal{H}(x')\}_{\mathrm{PB}} + \frac{3}{4} i\hbar^3 \left(\frac{\delta\mathcal{H}(x)}{\delta A^2 \delta E} \frac{\delta\mathcal{H}(x')}{\delta E^2 \delta A} - (x \leftrightarrow x') \right) \tag{171}$$

$$\big[\mathcal{H}(x), \mathcal{D}_i(x')\big]_M \quad = \quad i\hbar\{\mathcal{H}(x), \mathcal{D}_i(x')\}_{\mathrm{PB}} - \frac{3}{4} i\hbar^3 \frac{\delta^3\mathcal{H}(x)}{\delta E^2 \delta A} \frac{\delta^3\mathcal{D}_i(x')}{\delta A^2 \delta E} \tag{172}$$

where we have suppressed the indices on the A, E. An explicit and straightforward computation gives

$$\begin{aligned}
\omega_3(\mathcal{H}(x), \mathcal{H}(x')) &= -12i\delta(x,x')\left(E^j_a(x)A^a_j(x) - E^j_a(x')A^a_j(x')\right) && (173)\\
\omega_3(\mathcal{H}(x), \mathcal{D}_m(x')) &= 9i\delta_{,m}(x,x') && (174)
\end{aligned}$$

We notice that the first of these vanish in the sense of distributions, hence the only quantum correction is the constant (w.r.t. the phasespace variables) $9i\delta_{,m}(x,x')$. Consequently, the anomalous nature of gravity shows itself in the Ashtekar variables simply in a central extension of the constraint algebra (similar to a Schwinger term in current algebra).

$$[\mathcal{H}(x), \mathcal{D}_i(x')]_M = i\hbar\{\mathcal{H}(x), \mathcal{D}_i(x')\}_{\mathrm{PB}} - 9i\hbar^3\delta_i(x,x') \tag{175}$$

As we have seen earlier such central extensions can be "lifted" by means of a redefinition of the quantum constraints involving negative powers of $\hbar$.
Furthermore, since the constraints are polynomial in the phasespace variables the equations defining the physical state space, $\tilde{\mathcal{C}}_{\mathrm{phys}}$, become finite order differential equations. Explicitly, since the constraints are at most quartic in the phasespace variables

$$\begin{aligned}
0 = [\mathcal{H}, W]^+_M &= 2\mathcal{H}W - \frac{1}{2}\hbar^2\left(E^k_b E^l_v \epsilon_a{}^{bc}\epsilon^a{}_{ef}\frac{\delta^2 W}{\delta E^k_e \delta E^l_f} - \right.\\
&\quad 2\epsilon_a{}^{bc}\left(-\delta^a_e(\delta^k_i\partial_j - \delta^k_j\partial_i) + \epsilon^a{}_{pq}(\delta^p_e\delta^k_i A^q_j + \delta^q_e\delta^k_j A^p_i)\right)(\delta^i_l\delta^b_f E^j_c + \delta^j_l\delta^c_f E^i_b)\frac{\delta^2 W}{\delta E^k_e \delta A^f_l} + \\
&\quad \left.\epsilon_a{}^{bc}F^a_{ij}\frac{\delta^2 W}{\delta A^b_i \delta A^c_j}\right) + \frac{5}{4}\hbar^4\epsilon_{bc}{}^a\epsilon_a{}^{ef}\frac{\delta^4 W}{\delta E^k_b \delta E^l_c \delta A^e_k \delta A^f_l} && (176)\\
0 = [\mathcal{D}_i, W]^+_M &= 2\mathcal{D}_i W - \frac{1}{2}\hbar^2\left(\epsilon^a{}_{ef}E^j_a\frac{\delta^2 W}{\delta E^i_e \delta E^j_f} - \right.\\
&\quad \left. 2\left(-\delta^a_e(\delta^k_i\partial_j - \delta^k_j\delta_i) + \epsilon^a{}_{mn}(\delta^m_e\delta^k_i A^m_j + \delta^n_e\delta^k_j A^m_i)\right)\frac{\delta^2 W}{\delta E^k_e \delta A^a_j}\right. && (177)\\
0 = [\mathcal{G}_a, W]^+_M &= 2\mathcal{G}_a W + \frac{1}{4}i\hbar^2\delta^j_k\epsilon^c{}_{ab}\frac{\delta^2 W}{\delta A^c_j \delta A^k_b} && (178)
\end{aligned}$$

These coupled equations constitute the equations for the Wigner function for Ashtekar gravity in vacuum.
So far the Ashtekar variables have turned out to be much more tractable than the ADM-approach; the "anomaly" was simply a Schwinger term and the equations for the physical states are finite order. There is one problem, however, which one

seldom take into account in this formalism, namely the reality conditions. The physical variables have to correspond to a real metric, not a complex one. This condition is a second class constraint, [17], and this is where the real problems with the Ashtekar formalism lies. But since we have seen that second class constraints can be treated rather easily in this formalism, the conclusion must be that the Ashtekar approach to gravity, all things considered, is the most fruitful one.
As a final comment, let us digress on the recently proposed "Kuchař algebra", [14]. Here the Hamiltonian constraint $\mathcal{H}_\perp$ of the ADM approach is replaced by an abelian constraint $\mathcal{K}$ (a scalar density of weight ω), leading to the algebra of $C^\infty(\Sigma) \ltimes \mathrm{Diff}(\Sigma)$, i.e.,

$$\begin{aligned}
\{\mathcal{K}(x), \mathcal{K}(x')\}_{\mathrm{PB}} &= 0 && (179) \\
\{\mathcal{K}(x), \mathcal{H}_i(x')\}_{\mathrm{PB}} &= \mathcal{K}_{,i}\delta(x,x') + \omega\mathcal{K}\delta_{,i}(x,x') && (180) \\
\{\mathcal{H}_i(x), \mathcal{H}_j(x')\}_{\mathrm{PB}} &= \mathcal{H}_i(x)\delta_{,j}(x,x') + \mathcal{H}_j(x')\delta_{,i}(x,x') && (181)
\end{aligned}$$

There is an appealing interpretation of this algebra in terms of fibrebundles over a three-manifold, [15], but since the only known representations of this algebra are the ones formed from the ADM constraints, such as

$$\mathcal{K} = \mathcal{H}_\perp^2 - \mathcal{H}_i\mathcal{H}_j g^{ij}$$

and even more non-linear expressions (the general solution being found in [14]), we will not be able to compute the Moyal brackets in any satisfactory way – they will *a priori* involve infinite order differentials. There is, however, still the hope that some of the solutions, i.e., some of the explicit expressions for the $\mathcal{K}$'s, will turn out to simplify the Moyal bracket and will hence be somehow favoured. At the present, it has to be admitted, though, that this is a rather faint hope.
It is not enough for a constraint algebra to be nice, it is also important for the representation of the constraint algebra as functions on phasespace to be sufficiently simple, in order for the deformation quantisation scheme to be really tractable. This of course holds in practice for any quantisation scheme, but contrary to some other schemes, deformation quantisation is at least in principle able to handle constraints of arbitrary complecity – it is merely a matter of computational convenience (or laziness).
In a sequel paper we will concentrate on the deformation quantisation of gravity, and we will interpret the set of quantum constraints in the Ashtekar variables, find the relationship to the loop formalism and knot invariants and finally find an explicit solution related to topological field theory, [28].

5 Underlying Geometrical Structure

In the present picture one quantizes a classical theory, as described by an algebra of observables $\mathcal{A}_0 = C^\infty(\Gamma)$, by deforming it to obtain a noncommutative algebra $\mathcal{A}_\hbar \simeq \mathcal{A}_0 \otimes \mathbb{C}((\hbar))$ (isomorphism as vectorspaces, not as algebras), with a new, twisted, product $f, g \mapsto f * g = fg + O(\hbar)$.

We can consider $\mathcal{A}_\hbar$ as a "field" (in the language of C^*-algebras) over the real axis, parametrised by $\hbar \in \mathbb{R}$. This is the point of view taken by Nest et al, [24]. Alternatively, we can consider a *sheaf*, [29, 30], $\mathcal{A}$ over the topological space $X = \mathbb{R}$, where the stalks are given by

$$\mathcal{A}_x = \begin{cases} 0 & x < 0 \\ \mathcal{A}_0 & x = 0 \\ \mathcal{A}_\hbar & x > 0 \end{cases} \tag{182}$$

where $\hbar$ is taken to be a free parameter, $\hbar := x$. This is obviously a fine sheaf, since a partition of the identity trivially exists, furthermore, it is a sheaf which is almost constant. This expresses quantisation as a sheafication process.

We have already commented a bit on some underlying cohomological structure, here we will elaborate on this. First, each $f \in \mathcal{A}_\hbar$ can be written as a Laurent series (we can without loss of generality assume only one negative power of $\hbar$ to be present)

$$f = f_{-1}\hbar^{-1} + f_0 + f_1\hbar + f_2\hbar^2 + ... \qquad f_i \in \mathcal{A}_0 \tag{183}$$

This leads to the definition of a series of natural differentials, each with their own interpretation

$$\begin{aligned} \delta_0 &: f \to f_0 \\ \delta_q &: f \to f_1\hbar + \hbar^2 f_2 + ... \\ \delta_- &: f \to f_{-1} \\ \delta_+ &: f \to f_1 \end{aligned}$$

where $\delta_\pm$ are inspired by the corresponding definition for the Weil complex, [24]. Clearly $\delta_0^2 = \delta_+^2 = \delta_-^2 = 0$ so these three all define cohomology theories by considering the trivial complex

$$0 \to \mathcal{A}_\hbar \xrightarrow{\delta} \mathcal{A}_\hbar \xrightarrow{\delta} \mathcal{A}_\hbar \to ... \tag{184}$$

where δ denotes any of the differentials $\delta_\pm, \delta_0$. The cohomologies are trivial

$$H^0(\delta_0) = \mathcal{A}_0 \qquad H^0(\delta_\pm) = \delta_\pm \mathcal{A}_\hbar \tag{185}$$

thus, $H^0(\delta_0)$ gives the classical algebra back, whereas $H^0(\delta_-)$ gives the anomalous part.
A more interesting complex is

$$0 \to \mathcal{A}_\hbar \stackrel{\delta_0}{\to} \mathcal{A}_\hbar \stackrel{\delta_q}{\to} \mathcal{A}_\hbar \to ... \tag{186}$$

with alternating δ_0, δ_q. This is indeed a complex since obviously $\delta_0\delta_q + \delta_q\delta_0 = 0$. This has the same cohomology as the δ_0-complex, since $\mathrm{Im}\delta_0 = \mathrm{Ker}\delta_q$.
But, instead of considering the cohomology of a single stalk, it is better to consider the cohomology of the entire sheaf. This will also allow us to take the physicality condition into account. This condition can be rewritten as

$$sf := [\eta^\alpha \Phi_\alpha, f]_M = 0 \tag{187}$$

where now η^α a constant Grassmann numbers and the Moyal bracket is graded appropriatly - when f is Grassmann even this becomes the anti-Moyal bracket. Then, $s^2 = 0$, and s is very similar to the BRST differential. By extending the algebras $\mathcal{A}_x$ to include the constant Grassmann parameters η^α, we get a complex of sheaves

$$s : \mathcal{A}^q \to \mathcal{A}^{q+1} \tag{188}$$

where $\mathcal{A}^q$ denotes the sheaf obtained from $\mathcal{A}$ by tensoring

$$\mathcal{A}^q = \mathcal{A} \otimes \underbrace{\mathbb{G} \otimes \mathbb{G} \otimes ... \otimes \mathbb{G}}_{q} \tag{189}$$

where $\mathbb{G}$ denotes the space of the η_α-parameters. This leads us naturally to the concept of *hypercohomology*, [30]. Let $\{I\}$ be a covering of $X = \mathbb{R}$ by open sets, i.e., $\{I\}$ consists of open interval $]a, b[$ such that their union gives the entire real axis. Given such a covering we can define Čech cochains with values in the sheaf complex $\mathcal{A}^*$, the set of these is denoted by $\check{C}^p(I, \mathcal{A}^*)$ and is defined by, [29],

$$\check{C}^p(I, \mathcal{A}^*) = \{f : I^{p+1} \to \mathcal{A}^*\} \tag{190}$$

The Čech cochains come with a differential, $\check{\delta}$, given by

$$\check{\delta} f(I_0, ..., I_{p+1}) = f(I_0 \cap I_{p+1}, I_1, ..., I_p) + \sum_{i>j} (-1)^{i+j} f(I_0, ..., \hat{I}_i, ..., \hat{I}_j, ...I_i \cap I_j) \tag{191}$$

We hence have a bicomplex, $\check{C}^p(I, \mathcal{A}^q)$, the cohomology of the total complex is denoted by simply $H^*(I, \mathcal{A}^*)$. Now, any refinement of the cover induces homomorphisms among the cochains and hence also among the cohomology modules,

consequently, we can perform the limit of ever finer coverings to obtain the hypercohomology

$$\mathbb{H}^*(X, \mathcal{A}^*) := \lim_I H^*(I, \mathcal{A}^*) \tag{192}$$

Since the sheaf is fine, we have

$$\mathbb{H}^p(X, \mathcal{A}^q) = 0 \qquad p \neq 0 \tag{193}$$

On the other hand

$$\mathbb{H}^0(X, \mathcal{A}^q) = \lim_I \mathrm{Ker}(\check{\delta} + s) \tag{194}$$

In ghost number zero, we therefore have

$$\mathbb{H}^0(X, \mathcal{A}^0) = \mathcal{C}_{\mathrm{phys}} \tag{195}$$

This method is compatible with BRST-techniques. One simply defines the quantisation by means of the abovementioned sheafication procedure. This remedies the usual shortcoming of canonical quantisation techniques, such as the BRST, by avoiding finding an operator realisation. One just considers the BRST symmetry as a classical symmetry, and then deforms it by replacing the BRST-complex by its sheafication over the real axis.

6 Conclusion

We have studied the deformation (i.e., Moyal) quantisation of Hamiltonian systems with constraints. Instead of looking for an operator correspondence $q_i, p^i \to \hat{q}_i, \hat{p}^i$ with the usual rule $\{f, g\}_{\mathrm{PB}} \to \frac{1}{i\hbar}[\hat{f}, \hat{g}]$, we keep the classical phasespace but replace the Poisson bracket with the Moyal bracket, $[f, g]_M = i\hbar\{f, g\}_{\mathrm{PB}} + O(\hbar^2)$. It is know that this is always possible. By introducing sufficiently many phasespace variables and constraints one can assume the classical phasespace to be flat (i.e., to have a global patch of Darboux coordinates, $\{q_i, p^j\}_{\mathrm{PB}} = \delta_i^j$).
A priori one have to replace the classical constraints ϕ_a by quantum deformed versions $\Phi_a = \sum_n \hbar^n \Phi_a^{(n)}$ with $\Phi_a^{(0)} = \phi_a$ in order to keep the constraint algebra when one replaces Poisson brackets by Moyal ones. We saw that for Yang-Mills systems we could take $\Phi_a = \phi_a$, but for systems where the constraints are more than cubic in the phasespace variables, one would in general have $\Phi_a \neq \phi_a$.
Second class constraints could be handled by a similar device, turning them into a an Abelian algebra under Moyal brackets. This was illustrated by the case of the $O(N)$ non-linear σ-model.
The dynamics of a constrained Hamiltonian system was captured by the Wigner function, where the classical condition $\phi_a = 0$ was replaced not by the operator

equation $\hat{\phi}_a|\Psi\rangle = 0$ but by the algebraic condition $[\Phi_a, W]^+_M = 0$ resembling the BRST approach.
Various problems can arrise: (1) the appearance of second class constraints, (2) the appearance of first class constraints not in involution with the Hamiltonian, (3) the appearance of anomalies. It has been shown in this paper how all these problems can be remedied in deformation quantisation, provided one is willing to include negative powers of the deformation paramater $\hbar$ in the expansions of the quantum objects and to let the structure coefficients receive $\hbar$-corrections. Hence, upon replacing $\mathcal{A}_\hbar = \mathcal{A}\otimes\mathbb{C}[[\hbar]]$ ($\mathcal{A}$-valued formal power series in $\hbar$) by $\mathcal{A}_\hbar = \mathcal{A}\otimes\mathbb{C}((\hbar))$ (the set of $\mathcal{A}$-valued formal Laurent series in $\hbar$). This of course implies that the naive classical limit $\hbar \to 0$ becomes singular. The correct classical limit appears only upon taking principal values. Thus, the naive classical limit $A_{\mathrm{clas}} = \lim_{\hbar\to 0} A$ is replaced by

$$A_{\mathrm{clas}} = PP \lim_{\hbar\to 0} A = \delta_0 A \tag{196}$$

where PP denotes principal part and where δ_0 is the differential defined in the previous section.
With this definition the classical part is *still* the $\hbar^0$ component, there is just a subtle difference between picking out this component and taking the limit $\hbar \to 0$. With this *caveat*, deformation quantisation emerges as a very powerful quantisation scheme indeed. In particular, on should note the very explicit form the relations determining the $\hbar$-modification to the classical objects take – an explicit form allowing us in certain examples to arrive at an explicit solution to all orders in $\hbar$. Consequently, deformation quantisation is a very constructive approach to quantisation. The *caveat* about the principal part also illuminates the nature of the classical limit, and in particular how ill-behaved *classical* theories can arrise from well-behaved *quantum* ones due to the singularity of the limit. Alternatively, one can view this as illustrating the quantum "smearing out" of problems in the classical theory. With this in mind, it would be very interesting to study the application of deformation quantisation to quantum chaos.
Yang-Mills theory was then quantised in this scheme and we saw that in the absence of matter, the theory was anomaly free.
Gravitation was treated in three different manners, first the standard ADM approach, secondly in the Ashtekar variables and finally with a recently proposed new constraint algebra. We found that gravity in the ADM formalism was anomalous and lead to infinite order equations for the physical states – both problems stemming from the non-polynomial nature of the constraints. The Ashtekar variables, however, turned out to aquire merely a central extension, which can be lifted. Moreover, since these constraints were polynomial, the equations picking out physical states became finite order.

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